Sec 3.3:

1. Consider

$$A = \begin{bmatrix} 1 & 0 & 3 & -4 \\ -2 & 1 & -6 & 6 \\ 1 & 0 & 2 & -1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

Find the complete solution to $A\boldsymbol{x} = \boldsymbol{b}$.

<u>Particular Solution</u>

Solve $A\vec{x}_p = \vec{b}$ and set free variables to 0.

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & -4 & 1 \\ -2 & 1 & -6 & 6 & 3 \\ 1 & 0 & 2 & -1 & 2 \end{array}\right] \begin{array}{l} \\ R_2+2R_1 \Rightarrow \\ R_3-R_1 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 3 & -4 & 1 \\ 0 & 1 & 0 & -2 & 5 \\ 0 & 0 & -1 & 3 & 1 \end{array}\right] \begin{array}{l} \\ \\ -R_3 \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 3 & -4 & 1 \\ 0 & 1 & 0 & -2 & 5 \\ 0 & 0 & 1 & -3 & -1 \end{array}\right] \begin{array}{l} R_1-3R_3 \\ \Rightarrow \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 5 & 4 \\ 0 & 1 & 0 & -2 & 5 \\ 0 & 0 & 1 & -3 & -1 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1 + 5x_4 = 4 \Rightarrow x_1 = 4 \\ x_2 - 2x_4 = 5 \Rightarrow x_2 = 5 \\ x_3 - 3x_4 = -1 \Rightarrow x_3 = -1 \\ x_4 = 0 \end{cases} \quad \Rightarrow \vec{x}_p = \begin{bmatrix} 4 \\ 5 \\ -1 \\ 0 \end{bmatrix}$$

<u>Nullspace Solution</u>

Solve $A\vec{x}_n = \vec{0}$

$$A \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{bmatrix}, \text{ so } \begin{cases} x_1 + 5x_4 = 0 \Rightarrow x_1 = -5x_4 \\ x_2 - 2x_4 = 0 \Rightarrow x_2 = 2x_4 \\ x_3 - 3x_4 = 0 \Rightarrow x_3 = 3x_4 \\ x_4 \text{ free} \end{cases} \Rightarrow \vec{x}_n = x_4 \begin{bmatrix} -5 \\ 2 \\ 3 \\ 1 \end{bmatrix}.$$

$$\vec{x} = \vec{x}_p + \vec{x}_n = \begin{bmatrix} 4 \\ 5 \\ -1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

2. Find the complete solution of the system

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}.$$

Particular Solution

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 1 & 3 & 1 & 6 & 7 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 4 & 6 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1 + 3x_2 + 2x_4 = 1 \Rightarrow x_1 = 1 \\ x_2 = 0 \\ x_3 + 4x_4 = 6 \Rightarrow x_3 = 6 \\ x_4 = 0 \end{cases} \Rightarrow \vec{x}_p = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix}$$

Nullspace Solution

$$A \to \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } \begin{cases} x_1 + 3x_2 + 2x_4 = 0 \Rightarrow x_1 = -3x_2 - 2x_4 \\ x_2 \text{ free} \\ x_3 + 4x_4 = 0 \Rightarrow x_3 = -4x_4 \\ x_4 \text{ free} \end{cases} \Rightarrow \vec{x}_n = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

$$\vec{x} = \vec{x}_p + \vec{x}_n = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

3. Under what condition on b_1, b_2, b_3 is the following system solvable?

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= b_1 \\ 2x_1 + 5x_2 - 4x_3 &= b_2 \\ 4x_1 + 9x_2 - 8x_3 &= b_3 \end{aligned}$$

Solvable if $\vec{b} \in Col(A)$.

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-4R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2-2b_1 \\ 0 & 1 & 0 & b_3-4b_1 \end{array}\right] \begin{array}{l} \\ \\ R_3-R_2 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2-2b_1 \\ 0 & 0 & 0 & b_3-b_2-2b_1 \end{array}\right]$$

Solvable if $b_3 - b_2 - 2b_1 = 0$.

4. Choose the number q so that (if possible) the ranks of A and B are (i) 1, (ii) 2, (iii) 3:

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}.$$

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix} \xrightarrow{\frac{1}{6}R_1} \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix} \begin{matrix} R_2+3R_1 \\ R_3-9R_1 \end{matrix} \rightarrow \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & q-3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & q-3 \\ 0 & 0 & 0 \end{bmatrix}$$

i) rank(A) = 1 if $q-3=0 \Rightarrow q=3$.
ii) rank(A) = 2 if $q-3 \neq 0 \Rightarrow q \neq 3$.
iii) We can't have rank(A) = 3.

$$B = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & \frac{1}{3} & 1 \\ q & 2 & q \end{bmatrix} R_2 - qR_1 \rightarrow \begin{bmatrix} 1 & \frac{1}{3} & 1 \\ 0 & 2-\frac{q}{3} & 0 \end{bmatrix}$$

i) rank(B) = 1 if $2-\frac{q}{3}=0 \Rightarrow \frac{q}{3}=2 \Rightarrow q=6$.
ii) rank(B) = 2 if $2-\frac{q}{3} \neq 0 \Rightarrow q \neq 6$.
iii) We can't have rank(B) = 3.

Sec 3.4:

1. Consider the following subspaces Y, W, and V.

   (a) Describe each subspace (line, plane, etc).

   (b) Determine which of these form a **basis** for the subspace? Why or why not?

   (c) What is the **dimension** of each subspace?

   i.
$$Y = \text{span}\left\{ \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$$

a) Y is a plane in $\mathbb{R}^3$.

b) Basis is $\left\{ \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

c) $\dim(Y) = 2$.

   ii.
$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \end{bmatrix}, \right\}$$

a) W spans $\mathbb{R}^2$, since the span contains 2 LI vectors.

b) Any two of these vectors are a basis, e.g., $\left\{ \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$.

c) $\dim W = 2$.

   iii.
$$V = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 1 \\ 12 \end{bmatrix} \right\}$$

a) V is a plane in $\mathbb{R}^4$, since the span contains 2 LI vectors.

b) Any two of these vectors is a basis, e.g., $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -1 \\ 2 \end{bmatrix} \right\}$.

2. Consider the following matrix-vector system

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 4 & 8 \\ 4 & 8 & 6 & 8 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 4 \\ 2 \\ 10 \end{bmatrix}$$

(a) Find a **basis** for the column space of the matrix A. What is the rank of A?

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 4 & 8 \\ 4 & 8 & 6 & 8 \end{bmatrix} \begin{matrix} \\ R_2-2R_1 \\ R_3-4R_1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 2 & 8 \end{bmatrix} \begin{matrix} \\ \\ R_3-R_2 \end{matrix} \Rightarrow \begin{array}{c} \begin{matrix} \text{pivot} & & \text{pivot} & \end{matrix} \\ \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} = REF(A).$$

So a basis for $Col(A)$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} \right\}$, and $\dim(Col(A)) = 2$.

(b) Find a **basis** for the row space of the matrix A.

A basis for $row(A) = col(A^T)$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 4 \\ 8 \end{bmatrix} \right\}$.

(c) Find a **basis** for the null space of the matrix A.

$$A\vec{x} = \vec{0} \Rightarrow \begin{cases} x_1 + 2x_2 + x_3 = 0 \Rightarrow x_1 = -2x_2 + 4x_4 \\ x_2 \text{ free} \\ 2x_3 + 8x_4 = 0 \Rightarrow x_3 = -4x_4 \\ x_4 \text{ free} \end{cases} \Rightarrow \vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

So a basis for $Nul(A)$ is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$

(d) Find the complete solution $\boldsymbol{x} = \boldsymbol{x_p} + \boldsymbol{x_n}$.

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 4 \\ 2 & 4 & 4 & 8 & 2 \\ 4 & 8 & 6 & 8 & 10 \end{array}\right] \begin{matrix} \\ R_2-2R_1 \\ R_3-4R_1 \end{matrix} \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 2 & 8 & -6 \\ 0 & 0 & 2 & 8 & -6 \end{array}\right] \begin{matrix} \\ \\ R_3-R_2 \end{matrix} \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 2 & 8 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 + 2x_2 + x_3 = 4 \Rightarrow x_1 = 7 \\ x_2 = 0 \\ 2x_3 + 8x_4 = -6 \Rightarrow x_3 = -3 \\ x_4 = 0 \end{cases}$$

$$\vec{x}_p = \begin{bmatrix} 7 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \text{ so } \vec{x} = \vec{x}_p + \vec{x}_n = \begin{bmatrix} 7 \\ 0 \\ -3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

(e) Choose a different $\boldsymbol{x_p}$ from the one used in Part (a). Show that $\boldsymbol{x} = \boldsymbol{x_p} + \boldsymbol{x_n}$, with the new $\boldsymbol{x_p}$, still solves the system.

Let $x_2 = 1$ and $x_4 = 0$.

$$\Rightarrow \begin{cases} x_1 + 2x_2 + x_3 = 4 \Rightarrow x_1 = 5 \\ x_2 = 1 \\ 2x_3 + 8x_4 = -6 \Rightarrow x_3 = -3 \\ x_4 = 0 \end{cases} \Rightarrow \vec{x}_p = \begin{bmatrix} 5 \\ 1 \\ -3 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 4 & 8 \\ 4 & 8 & 6 & 8 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 10 \end{bmatrix}$$

3. Consider the following matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix}$$

(a) What's in the nullspace of A?

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \Rightarrow \\ R_3 + 2R_1 \end{matrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}_{R_3 + R_2} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 + x_2 = 0 \Rightarrow x_1 = 0 \\ x_2 = 0 \end{cases}$$

$$\Rightarrow \mathrm{Nul}(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

(b) For any $\boldsymbol{b}$, how many solutions do you expect $A\boldsymbol{x} = \boldsymbol{b}$ to have?

A has more equations than vars, so there's either ∞ solutions or none.

(c) What is the condition on $\boldsymbol{b} = [b_1,\ b_2,\ b_3]$ such that $A\boldsymbol{x} = \boldsymbol{b}$ is solvable?

$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{array}\right] \begin{matrix} \\ R_2 - R_1 \Rightarrow \\ R_3 + 2R_1 \end{matrix} \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & -1 & b_3 + 2b_1 \end{array}\right]_{R_3 + R_2} \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 + b_2 + b_1 \end{array}\right].$$

$b_3 + b_2 + b_1 = 0.$

4. Consider the following matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

(a) If $\boldsymbol{b} = \begin{bmatrix} 3 \\ 6 \\ \beta \end{bmatrix}$, for what values of the scalar β will $A\boldsymbol{x} = \boldsymbol{b}$ have a solution?

$$\left[\begin{array}{ccccc|c} 0 & 1 & 2 & 3 & 4 & 3 \\ 0 & 1 & 2 & 4 & 6 & 6 \\ 0 & 0 & 0 & 1 & 2 & \beta \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccccc|c} 0 & 1 & 2 & 3 & 4 & 3 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 & \beta \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccccc|c} 0 & 1 & 2 & 3 & 4 & 3 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & \beta - 3 \end{array}\right]$$

$\beta - 3 = 0 \Rightarrow \beta = 3.$

(b) For the β from part (a), find the **complete** solution to $A\boldsymbol{x} = \boldsymbol{b}$.

$$\left[\begin{array}{ccccc|c} 0 & 1 & 2 & 3 & 4 & 3 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 0 & 1 & 2 & 3 & 4 & 3 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 \text{ free} \\ x_2 + 2x_3 + 3x_4 + 4x_5 = 3 \Rightarrow x_2 = -9 - 2x_3 + 2x_5 \\ x_3 \text{ free} \\ x_4 + 2x_5 = 3 \Rightarrow x_4 = 3 - 2x_5 \\ x_5 \text{ free} \end{cases}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ -9 - 2x_3 + 2x_5 \\ x_3 \\ 3 - 2x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$