Sec 3.1:

1. Which of the following subsets of $\mathbb{R}^3$ are subspaces? Justify your answer.

   (a) The plane of vectors (b_1, b_2, b_3) that satisfy $b_1 = 0$.

$B = \{\vec{b} \in \mathbb{R}^3 : b_1 = 0\}$

$\vec{0} \in B$

Let $\vec{u} = \begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ v_2 \\ v_3 \end{bmatrix}$, $a, b \in \mathbb{R}$

$$a\vec{u} + b\vec{v} = \begin{bmatrix} 0 \\ au_2 \\ au_3 \end{bmatrix} + \begin{bmatrix} 0 \\ bv_2 \\ bv_3 \end{bmatrix} = \begin{bmatrix} 0 \\ au_2 + bv_2 \\ au_3 + bv_3 \end{bmatrix} \in B$$

$\therefore B$ is a subspace

   (b) The plane of vectors (b_1, b_2, b_3) that satisfy $b_1 = 1$.

Let $V = \{\vec{v} \in \mathbb{R}^3 : v_1 = 1\}$

1) $0 \notin V$

2) $\begin{pmatrix} 1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} 1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 2 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix} \notin V$

3) $c \begin{pmatrix} 1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} c \\ cu_2 \\ cu_3 \end{pmatrix} \notin V$

By any one of these, V is not a subspace.

(c) All combinations of two given vectors $(1, 1, 0)$ and $(2, 0, 1)$.

$$\alpha\left(a\begin{bmatrix}1\\1\\0\end{bmatrix}+b\begin{bmatrix}2\\0\\1\end{bmatrix}\right)+\beta\left(c\begin{bmatrix}1\\1\\0\end{bmatrix}+d\begin{bmatrix}2\\0\\1\end{bmatrix}\right)$$

$$=\alpha a\begin{bmatrix}1\\1\\0\end{bmatrix}+\alpha b\begin{bmatrix}2\\0\\1\end{bmatrix}+\beta c\begin{bmatrix}1\\1\\0\end{bmatrix}+\beta d\begin{bmatrix}2\\0\\1\end{bmatrix}$$

$$=(\alpha a+\beta c)\begin{bmatrix}1\\1\\0\end{bmatrix}+(\alpha b+\beta d)\begin{bmatrix}2\\0\\1\end{bmatrix}$$

$\therefore$ subspace

(d) The plane of vectors (b_1, b_2, b_3) that satisfy $b_3 - b_2 + 3b_1 = 0$.

$U = \{\vec{u} \in \mathbb{R}^3 : u_3 - u_2 + 3u_1 = 0\}$

$0 - 0 + 3\cdot 0 = 0$, so $\vec{0} \in U$

Let $\vec{u}, \vec{v} \in U$.

Then $u_3 - u_2 + 3u_1 = v_3 - v_2 + 3v_1 = 0$

$$a\vec{u}+b\vec{v}=\begin{bmatrix}au_1+bv_1\\au_2+bv_2\\au_3+bv_3\end{bmatrix}$$

$au_1+bv_1-au_2-bv_2+3au_3+3bv_3$

$(au_1-au_2+3au_3)+(bv_1-bv_2+3bv_3)$

$a\underbrace{(u_1-u_2+3u_3)}_{=0}+b\underbrace{(v_1-v_2+3v_3)}_{=0}=0$

$\Rightarrow a\vec{u}+b\vec{v} \in U$

$\therefore U$ subspace

2. The set $W \subseteq \mathbb{R}^3$ is the set of all vectors $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ whose coordinates satisy

$$x_1 - x_2 + 2x_3 = 0$$

$$3x_2 - x_3 = 0$$

Determine if W is a subspace of $\mathbb{R}^3$.

$W = \{\vec{x} \in \mathbb{R}^3 : \vec{x} = t(-5, 1, 3), \ t \in \mathbb{R}\}$

$\vec{0} \in W$

$$a\left(t_1 \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}\right) + b\left(t_2 \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}\right)$$

$$(at_1 + bt_2)\begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix} \in W$$

$\therefore$ W is a subspace

3. (a) Show that the set of nonsingular 2×2 matrices is not a subspace.

Nonsingular $\Leftrightarrow$ invertible

1) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible

2) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3) $0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Any one of these shows it's not a subspace.

(b) Show also that the set of singular 2×2 matrices is not a subspace.

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{\text{singular}} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{singular}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{nonsingular}}$$

4. Describe the column spaces (lines or planes) of these particular matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$ Line

$$C(B) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\right\}$$ Plane

$$C(C) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right\}$$ Line

Sec 3.2:

1. Consider the following matrix

$$A = \begin{bmatrix} 1 & 0 & 3 & -4 \\ -2 & 1 & -6 & 6 \\ 1 & 0 & 2 & -1 \end{bmatrix}$$

(a) Perform elimination on A until upper triangular matrix appears.

$$A = \begin{bmatrix} 1 & 0 & 3 & -4 \\ -2 & 1 & -6 & 6 \\ 1 & 0 & 2 & -1 \end{bmatrix} \begin{matrix} \\ R_2+2R_1 \\ R_3-R_1 \end{matrix} \Rightarrow \begin{bmatrix} \boxed{1} & 0 & 3 & -4 \\ 0 & \boxed{1} & 0 & -2 \\ 0 & 0 & \boxed{-1} & 3 \end{bmatrix}$$

(b) Identify the pivot columns (put a square around the pivots) and the free columns (circle the whole column). What are the free variables? What are the pivot variables?

x_4 free

x_1, x_2, x_3 pivot

(c) Perform back substitution, writing the pivot variables in terms of the free variables.

$$\begin{cases} x_1 + 3x_3 - 4x_4 = 0 \Rightarrow x_1 = -9x_4 + 4x_4 = -5x_4 \\ x_2 - 2x_4 = 0 \Rightarrow x_2 = 2x_4 \\ -x_3 + 3x_4 = 0 \Rightarrow x_3 = 3x_4 \end{cases}$$

$$\vec{x} = \begin{bmatrix} -5x_4 \\ 2x_4 \\ 3x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -5 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

(d) Describe the nullspace of A.

$$\text{Nul}(A) = \text{span}\left\{ \begin{bmatrix} -5 \\ 2 \\ 3 \\ 1 \end{bmatrix} \right\}$$

2. Consider the following matrices

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}$$

(a) Compute REF(A) and REF(B). What are the ranks of A and B?

$$\begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix} \rightarrow \begin{bmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -2 \end{bmatrix} \begin{matrix} \\ \\ R_3 + R_2 \end{matrix} \rightarrow \begin{bmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \frac{1}{2}R_1 \\ \\ \\ \end{matrix}$$

Rank(A) = 2

$$\begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} R_2 - 3R_1 \rightarrow \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \tfrac{1}{2}R_2 \rightarrow \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

Rank(B) = 2

(b) Find Col$A$, and describe it geometrically.

$$\text{Col}(A) = \text{span}\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} \right\}$$ plane in $\mathbb{R}^3$

(c) Find Col$B$, and describe it geometrically.

$$\text{Col}(B) = \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix} \right\}$$ all of $\mathbb{R}^2$

(d) Compute RREF(A) and RREF(B).

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1-2R_2} \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix} \xrightarrow{R_1-2R_2} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

(e) Find Nul$A$, and describe it geometrically.

$$\begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 + x_3 - 2x_4 = 0 \\ x_2 + x_3 + 2x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -x_3 + 2x_4 \\ x_2 = -x_3 - 2x_4 \\ x_3 = \text{free} \\ x_4 = \text{free} \end{cases}$$

$$\vec{x} = \begin{bmatrix} -x_3 + 2x_4 \\ -x_3 - 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Nul}(A) = \text{span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{plane in } \mathbb{R}^4$$

(f) Find Nul$B$, and describe it geometrically.

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 + 2x_3 = 0 \Rightarrow x_1 = -2x_3 \\ x_2 + 2x_4 = 0 \Rightarrow x_2 = -2x_4 \end{cases} \Rightarrow \begin{cases} x_1 = -2x_3 \\ x_2 = -2x_4 \\ x_3 \text{ free} \\ x_4 \text{ free} \end{cases}$$

$$\vec{x} = \begin{bmatrix} -2x_3 \\ -2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Nul}(B) = \text{span}\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{plane in } \mathbb{R}^4$$