Sec 2.5:

1. Suppose A is invertible, and you exchange its first two rows to reach B.

   (a) Is the new matrix B invertible? Why?

   An $n \times n$ matrix is invertible IFF elimination produces n pivots (allowing row exchanges).
   Since A is invertible, $EPA = U$ where U has n pivots.
   Let P_{12} be the permutation matrix that exchanges row 1 and row 2.
   Then $B = P_{12}A \Rightarrow P_{12}^{-1}B = P_{12}^{-1}P_{12}A = A \Rightarrow P_{12}B = A$, since $P_{12}^{-1} = P_{12}$.
   $EPA = U \Rightarrow EPP_{12}B = U$. Letting $P' = PP_{12}$, we have $EP'B = U$ where U has n pivots.
   Therefore, B is invertible.

   (b) How would you find B^{-1} from A^{-1}?

   $B = P_{12}A$, so $B^{-1} = (P_{12}A)^{-1} = A^{-1}P_{12}^{-1} = A^{-1}P_{12}$.

2. If the product $M = ABC$ of three matrices is invertible, then A, B, and C are invertible. Find a formula for B^{-1} that involves M^{-1}, A, and C.

   $M = ABC$
   $A^{-1}M = A^{-1}ABC = BC$
   $A^{-1}MC^{-1} = BCC^{-1} = B$
   $B^{-1} = (A^{-1}MC^{-1})^{-1}$
   $= (C^{-1})^{-1}M^{-1}(A^{-1})^{-1}$
   $= CM^{-1}A.$

3. For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

A is invertible if elimination produces n pivots.

$$\begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix} \begin{matrix} R_2 - \frac{c}{2}R_1 \\ R_3 - 4R_1 \end{matrix} \rightarrow \begin{bmatrix} 2 & c & c \\ 0 & c-\frac{c^2}{2} & c-\frac{c^2}{2} \\ 0 & 7-4c & -3c \end{bmatrix} R_3 - \frac{7-4c}{c-\frac{c^2}{2}}R_2 \rightarrow \begin{bmatrix} 2 & c & c \\ 0 & c-\frac{c^2}{2} & c-\frac{c^2}{2} \\ 0 & 0 & c-7 \end{bmatrix}$$

So $c - \frac{c^2}{2} \neq 0$ and $c - 7 \neq 0$.
$c - 7 \neq 0 \Rightarrow c \neq 7.$
$c - \frac{c^2}{2} \neq 0 \Rightarrow c(1 - \frac{c}{2}) \neq 0 \Rightarrow c \neq 0, c \neq 2.$

4. Find the inverse using Gauss-Jordan elimination:

$[A|I] \rightarrow [I|A^{-1}]$

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \\ -2 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 1 & 0 \\ -2 & 0 & 2 & 0 & 0 & 1 \end{array}\right] R_3+R_1 \rightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right] R_3-\tfrac{1}{3}R_2 \rightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 1 & 0 \\ 0 & 0 & \frac{2}{3} & 1 & -\frac{1}{3} & 1 \end{array}\right] \begin{array}{l} \frac{1}{2}R_1 \\ \frac{1}{3}R_2 \\ \frac{3}{2}R_3 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \end{array}\right] \begin{array}{l} R_1+\frac{1}{2}R_3 \\ R_2-\frac{1}{3}R_3 \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{5}{4} & -\frac{1}{4} & \frac{3}{4} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \end{array}\right] R_1-\tfrac{1}{2}R_2 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & 1 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] R_2+\tfrac{1}{2}R_1 \rightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] R_3+\tfrac{2}{5}R_2 \rightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{7}{5} & \frac{1}{5} & \frac{2}{5} & 1 \end{array}\right] \begin{array}{l} \frac{1}{2}R_1 \\ \frac{2}{5}R_2 \\ \frac{5}{7}R_3 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 0 & 1 & \frac{1}{7} & \frac{2}{7} & \frac{5}{7} \end{array}\right] \begin{array}{l} R_1+\frac{1}{2}R_3 \\ R_2+\frac{3}{5}R_3 \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{4}{7} & \frac{1}{7} & \frac{5}{14} \\ 0 & 1 & 0 & \frac{2}{7} & \frac{4}{7} & \frac{3}{7} \\ 0 & 0 & 1 & \frac{1}{7} & \frac{2}{7} & \frac{5}{7} \end{array}\right] R_1-\tfrac{1}{2}R_2 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{7} & -\frac{1}{7} & \frac{1}{7} \\ 0 & 1 & 0 & \frac{2}{7} & \frac{4}{7} & \frac{3}{7} \\ 0 & 0 & 1 & \frac{1}{7} & \frac{2}{7} & \frac{5}{7} \end{array}\right]$$

$$B^{-1} = \begin{bmatrix} \frac{3}{7} & -\frac{1}{7} & \frac{1}{7} \\ \frac{2}{7} & \frac{4}{7} & \frac{3}{7} \\ \frac{1}{7} & \frac{2}{7} & \frac{5}{7} \end{bmatrix}$$

Sec 2.6, 2.7:

1. **LU Factorization.** Complete the following steps to find the LU factorization of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 1 \end{bmatrix}$$

(a) Find the two elimination matrices, E_{21}, E_{32} that will put A into upper-triangular form.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -5 \end{bmatrix} = U.$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

(b) Find E_{21}^{-1} and E_{32}^{-1} and multiply them in the correct order to find L such that $A = LU$.

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$E_{32}E_{21}A = U \Rightarrow E_{21}A = E_{32}^{-1}U \Rightarrow A = \overbrace{E_{21}^{-1}E_{32}^{-1}}^{L}U$$

$$L = E_{21}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

(c) Find the diagonal matrix D such that $A = LDU$, where both L and U have ones on their diagonals.

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -5 \end{bmatrix}$$

$$A = LDU' = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

(d) Check that your factorization works! Multiply your L, D, and U matrices to recover A.

$$A = LDU' = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 4 & -5 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 1 \end{bmatrix} = A$$

2. Consider the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}$$

(a) Compute the LU-factorization to find matrices L and U such that $A = LU$.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow{R_3 + 2R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & -5 \end{bmatrix} = U$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \Rightarrow E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$L = (E_{32}E_{21})^{-1} = E_{21}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & -5 \end{bmatrix}$$

(b) Switch rows 2 and 3 of A to find S. Show that this new matrix S is symmetric.

$$S = P_{23}A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$S^T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & 1 \\ 2 & 1 & 1 \end{bmatrix} = S, \text{ so } S \text{ is symmetric.}$$

(c) Compute the factorization $S = LDL^T$.

$$S = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & 1 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow{R_3 + \frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -\frac{5}{2} \end{bmatrix} = DL^T$$

$$DL^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$S = LDL^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

3. What three matrices E_{21}, E_{12} and D reduce $A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$ to the identity matrix (i.e., $DE_{12}E_{21}A = I$)? Multiply these matrices together to find A^{-1}.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$$E_{21} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$DE_{12}E_{21}A = I$

$$A^{-1} = DE_{12}E_{21} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ -1 & \frac{1}{2} \end{bmatrix} = A^{-1}.$$

4. Solve the system $A\boldsymbol{x} = \boldsymbol{b}$ using the LU factorization of A, where

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 8 & 9 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}.$$

(a) First factor A into LU, and obtain the system $LU\boldsymbol{x} = \boldsymbol{b}$

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 8 & 9 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{R_3 + \frac{1}{5}R_2} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & \frac{7}{5} \end{bmatrix} = U.$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{5} & 1 \end{bmatrix}$$

$$L = (E_{32}E_{31})^{-1} = E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{5} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -\frac{1}{5} & 1 \end{bmatrix}$$

$$\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -\frac{1}{5} & 1 \end{bmatrix}}^{L}\underbrace{\overbrace{\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & \frac{7}{5} \end{bmatrix}}^{U}\overbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}^{\vec{x}}}_{\vec{y}} = \overbrace{\begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}}^{\vec{b}}$$

(b) Use your result in (a) to compute $L\boldsymbol{y} = \boldsymbol{b}$. What is $\boldsymbol{y}$?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -\frac{1}{5} & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

$$\begin{cases} y_1 = 2 \\ y_2 = 2 \\ 3y_1 - \frac{1}{5}y_2 + y_3 = 5 \Rightarrow y_3 = -\frac{3}{5} \end{cases}$$

$$\vec{y} = \begin{bmatrix} 2 \\ 2 \\ -\frac{3}{5} \end{bmatrix}$$

(c) Use your result in (b) to find $\boldsymbol{x}$.

$U\vec{x} = \vec{y}$

$$\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & \frac{7}{5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -\frac{3}{5} \end{bmatrix}$$

$$\begin{cases} 2x_1 + 3x_2 + 3x_3 = 2 \Rightarrow x_1 = \frac{1}{7} \\ 5x_2 + 7x_3 = 2 \Rightarrow x_2 = 1 \\ \frac{7}{5}x_3 = -\frac{3}{5} \Rightarrow x_3 = -\frac{3}{7} \end{cases}$$

$$\vec{x} = \begin{bmatrix} \frac{1}{7} \\ 1 \\ -\frac{3}{7} \end{bmatrix}$$

(d) How could you factor A into a product UL, upper triangular times lower triangular? Would they be the same factors as in $A = LU$?

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 8 & 9 \end{bmatrix} \xrightarrow{R_2 - \frac{7}{9}R_3} \begin{bmatrix} 2 & 3 & 3 \\ -\frac{14}{3} & -\frac{11}{9} & 0 \\ 6 & 8 & 9 \end{bmatrix} \xrightarrow{R_1 - \frac{1}{3}R_3} \begin{bmatrix} 0 & \frac{1}{3} & 0 \\ -\frac{14}{3} & -\frac{11}{9} & 0 \\ 6 & 8 & 9 \end{bmatrix} \xrightarrow{R_1 + \frac{3}{11}R_2} \begin{bmatrix} -\frac{14}{11} & 0 & 0 \\ -\frac{14}{3} & -\frac{11}{9} & 0 \\ 6 & 8 & 9 \end{bmatrix} = L$$

$$E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{9} \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{13} = \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 1 & \frac{3}{11} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{12}E_{13}E_{23}A = L$$

$$U = (E_{12}E_{13}E_{23})^{-1} = E_{23}^{-1}E_{13}^{-1}E_{12}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{7}{9} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\frac{3}{11} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{11} & \frac{1}{3} \\ 0 & 1 & \frac{7}{9} \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = UL = \begin{bmatrix} 1 & -\frac{3}{11} & \frac{1}{3} \\ 0 & 1 & \frac{7}{9} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -\frac{14}{11} & 0 & 0 \\ -\frac{14}{3} & -\frac{11}{9} & 0 \\ 6 & 8 & 9 \end{bmatrix}$$