# Sec 2.1, 2.2:

1. Elimination: three equations, three unknowns. Consider the following system of equations

$$x + 2y - z = 1 \tag{1a}$$
$$2x - y + z = 3 \tag{1b}$$
$$3x + y - 2z = 4 \tag{1c}$$

(a) Eliminate $x$.

i. Find the pivot in Eq. (1a) and multipliers from Eqs. (1b) and (1c) that will eliminate the $x$ terms in Eqs. (1b) and (1c).

Pivot: 1

Multiplier from Eq. (1b): -2

Multiplier from Eq. (1c): -3

ii. Perform the elimination of $x$ from the last two equations.

$$\begin{array}{rl} & 2x - y + z = 3 \qquad (1b) \\ + & -2x - 4y + 2z = -2 \qquad -2\cdot(1a) \\ \hline & -5y + 3z = 1 \end{array}$$

$$\begin{array}{rl} & 3x + y - 2z = 4 \qquad (1c) \\ + & -3x - 6y + 3z = -3 \qquad -3\cdot(1a) \\ \hline & -5y + z = 1 \end{array}$$

iii. Write the resulting 2D system of equations (with just $y$ and $z$) in which $x$ is eliminated. *Hint:* Leave Eq. (1a) out.

$$\begin{cases} -5y + 3z = 1 \\ -5y + z = 1 \end{cases}$$

(b) Eliminate $y$. From answer to (a)iii, find the pivot and multiplier to eliminate $y$. Perform the elimination and write the final equation that just contains $z$.

$$\begin{array}{r} \left\{\begin{array}{l} -5y+3z=1 \\ 5y-z=-1 \end{array}\right. \\ + \quad \\ \hline 2z=0 \end{array}$$

(c) Back substitution. Now, write the original Eq. (1a), the x-eliminated Eq. (1b) (that just contains $y$ and $z$) and the x,y-eliminated Eq. (1c) (just containing $z$). You should see a triangular system. Use back substitution to solve the system for $z$, then $y$, then $x$.

$$\left\{\begin{array}{l} x+2y-z=1 \\ -5y+3z=1 \\ 2z=0 \end{array}\right.$$

$2z=0 \Rightarrow z=0$

$-5y+3(0)=1 \Rightarrow y=-\frac{1}{5}$

$x+2(-\frac{1}{5})-0=1 \Rightarrow x=\frac{7}{5}$

2. Use Gauss-Jordan elimination to solve the following system:

$$2x_1 + 4x_2 - 2x_3 = 2 \tag{2a}$$
$$4x_1 + 9x_2 - 3x_3 = 8 \tag{2b}$$
$$-2x_1 - 3x_2 + 7x_3 = 4 \tag{2c}$$

$$+\begin{cases} 4x_1 + 9x_2 - 3x_3 = 8 & (2b) \\ -4x_1 - 8x_2 + 4x_3 = -4 & -2\cdot(2a) \end{cases}$$
$$x_2 + x_3 = 4 \qquad (2d)$$

$$+\begin{cases} -2x_1 - 3x_2 + 7x_3 = 4 & (2c) \\ 2x_1 + 4x_2 - 2x_3 = 2 & (2a) \end{cases}$$
$$x_2 + 5x_3 = 6 \qquad (2e)$$

$$+\begin{cases} x_2 + x_3 = 4 & (2d) \\ -x_2 - 5x_3 = -6 & -(2e) \end{cases}$$
$$-4x_3 = -2 \qquad (2f)$$

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 & (2a) \\ x_2 + x_3 = 4 & (2d) \\ -4x_3 = -2 & (2f) \end{cases}$$

$-4x_3 = -2 \Rightarrow x_3 = \frac{1}{2}$

$x_2 + \frac{1}{2} = 4 \Rightarrow x_2 = \frac{7}{2}$

$2x_1 + 4(\frac{7}{2}) - 2(\frac{1}{2}) \Rightarrow x_1 = -\frac{11}{2}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{11}{2} \\ \frac{7}{2} \\ \frac{1}{2} \end{bmatrix}$$

3. Are the vectors $\boldsymbol{u} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$, and $\boldsymbol{w} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ linearly independent?

*Hint*: Solve $c_1\boldsymbol{u} + c_2\boldsymbol{v} + c_3\boldsymbol{w} = \boldsymbol{0}$.

$$c_1\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} c_1 + c_2 + c_3 = 0 & \text{(a)} \\ 3c_1 + 3c_2 - c_3 = 0 & \text{(b)} \\ c_1 - c_2 + c_3 = 0 & \text{(c)} \end{cases}$$

$$\begin{array}{rl} 3c_1 + 3c_2 - c_3 = 0 & \text{(b)} \\ + \quad -3c_1 - 3c_2 - 3c_3 = 0 & -3\text{(a)} \\ \hline -4c_3 = 0 & \text{(d)} \end{array}$$

$$\begin{array}{rl} c_1 - c_2 + c_3 = 0 & \text{(c)} \\ + \quad -c_1 - c_2 - c_3 = 0 & -\text{(a)} \\ \hline -2c_2 = 0 & \text{(e)} \end{array}$$

$$\begin{cases} c_1 + c_2 + c_3 = 0 & \text{(a)} \\ -4c_3 = 0 & \text{(d)} \\ -2c_2 = 0 & \text{(e)} \end{cases}$$

$-4c_3 = 0 \Rightarrow c_3 = 0$

$-2c_2 = 0 \Rightarrow c_2 = 0$

$c_1 + 0 + 0 = 0 \Rightarrow c_1 = 0$

$c_1 = c_2 = c_3 = 0$, so $\{\vec{u}, \vec{v}, \vec{w}\}$ is LI.

4. Use elimination to solve

$$x_1 + x_2 + x_3 = 6 \tag{3a}$$
$$x_1 + 2x_2 + 2x_3 = 11 \tag{3b}$$
$$2x_1 + 3x_2 - 4x_3 = 3 \tag{3c}$$

$$+\begin{cases} x_1 + 2x_2 + 2x_3 = 11 & \text{(b)} \\ -x_1 - x_2 - x_3 = -6 & -\text{(a)} \end{cases}$$
$$x_2 + x_3 = 5 \quad \text{(d)}$$

$$+\begin{cases} 2x_1 + 3x_2 - 4x_3 = 3 & \text{(c)} \\ -2x_1 - 2x_2 - 2x_3 = -12 & -2\text{(a)} \end{cases}$$
$$x_2 - 6x_3 = -9 \quad \text{(e)}$$

$$+\begin{cases} x_2 + x_3 = 5 & \text{(d)} \\ -x_2 + 6x_3 = 9 & -\text{(e)} \end{cases}$$
$$7x_3 = 14 \quad \text{(f)}$$

$$\begin{cases} x_1 + x_2 + x_3 = 6 & \text{(a)} \\ x_2 + x_3 = 5 & \text{(d)} \\ 7x_3 = 14 & \text{(f)} \end{cases}$$

$7x_3 = 14 \Rightarrow x_3 = 2$

$x_2 + 2 = 5 \Rightarrow x_2 = 3$

$x_1 + 3 + 2 = 6 \Rightarrow x_1 = 1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

## Sec 2.3, 2.4:

1. Write down the $3 \times 3$ matrices that produce these steps:

   (a) Subtracts 5 times row 1 from row 2.

   $$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   (b) Adds 7 times row 2 to row 3.

   $$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$$

   (c) Exchanges rows 1 and 2, then rows 2 and 3.

   $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

   (d) Subtracts row 1 from row 2, and then exchanges rows 2 and 3.

   $$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

   (e) Exchanges rows 2 and 3, and then subtracts row 1 from row 3.

   $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

2. Elimination matrices: Consider the following system of equations:

$$x + 2y - z = 1$$
$$2x - y + z = 3$$
$$3x + y - 2z = 4$$

We are going to go through the process of solving a linear system using elimination matrices.

(a) Write the system of equations as a matrix vector system: $A\boldsymbol{x} = \boldsymbol{b}$.

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

(b) Identify the pivot in the first column (circle it in your matrix $A$). Write the two elimination matrices $E_{21}$ and $E_{31}$ that, when applied to $A$, will give zeros in the necessary positions.

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

(c) Perform the multiplication $E_{31}E_{21}A$. Do you see zeros below the pivot?

$$E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 3 & 1 & -2 \end{bmatrix}$$

$$E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & -5 & 1 \end{bmatrix}$$

(d) Now move on to the second column. Identify the new pivot and circle it in your new matrix from Part (c) [it might be useful to re-write that matrix here]. *Hint:* recall that we are looking to write the system as an upper-triangular system. Therefore, we are looking to get a zero under all of the diagonal entries of $A$.

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & -5 & 1 \end{bmatrix}$$

(e) Write the elimination matrix $E_{32}$ that, when applied to the matrix from Part (c), will give a zero in the necessary position. Multiply this elimination matrix with your matrix from Part (c). Do you see an upper-triangular matrix?

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$E_{32}E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & -5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & 0 & -2 \end{bmatrix}$$

(f) Multiply out the three elimination matrices from Parts (c) and (e): $E = E_{32}E_{31}E_{21}$.

$$E = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

(g) Now that you have one matrix, $E$, that describes all elimination operations performed on $A$, we can solve the system by applying this matrix to both sides of the equation: $EA\boldsymbol{x} = E\boldsymbol{b}$. Write out this new matrix-vector system.

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(h) Find the solution $\boldsymbol{x}$ by performing back substitution on the system from Part (g). *Hint:* Convert back to a system of equations.

$$\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ -5x_2 + 3x_3 = 1 \\ -2x_3 = 0 \end{cases}$$

$-2x_3 = 0 \Rightarrow x_3 = 0$

$-5x_2 + 3 \cdot 0 = 1 \Rightarrow x_2 = -\frac{1}{5}$ $\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{7}{5} \\ -\frac{1}{5} \\ 0 \end{bmatrix}$

$x_1 + 2(-\frac{1}{5}) - 0 = 1 \Rightarrow x_1 = \frac{7}{5}$

(i) Check your answer by substituting the answer back into the original system in Part (a).

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} \frac{7}{5} \\ -\frac{1}{5} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \vec{b}$$

3. Which three elimination matrices put $A$ into triangular form $U$?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{21}A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \qquad E_{31}E_{21}A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \qquad U = E_{32}E_{31}E_{21}A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

4. Which elimination matrices put $A$ into triangular form $U$?

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_{21}A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_{32}E_{21}A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{3}{4} & 1 \end{bmatrix} \qquad E_{43}E_{32}E_{21}A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix} = U$$