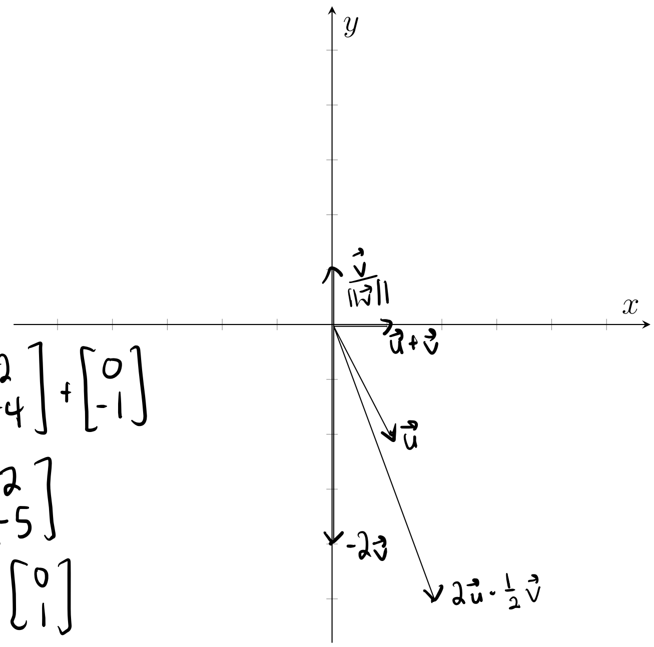
Sec 1.1, 1.2:

1. Consider the following two vectors:

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Draw and label the following vectors on the axis provided:

(a) $\boldsymbol{u}$



(b) $-2\boldsymbol{v} = -2\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$

(c) $\boldsymbol{u}+\boldsymbol{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(d) $2\boldsymbol{u} - 0.5\boldsymbol{v} = 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

(e) The *unit* vector, $\frac{\boldsymbol{v}}{||\boldsymbol{v}||}$

$\|\vec{v}\| = \sqrt{0^2+2^2} = 2$, so $\frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

What are the lengths of $\boldsymbol{u}$ and $\boldsymbol{v}$?

$\|\vec{u}\| = \sqrt{1^2+(-2)^2} = \sqrt{1+4} = 5.$

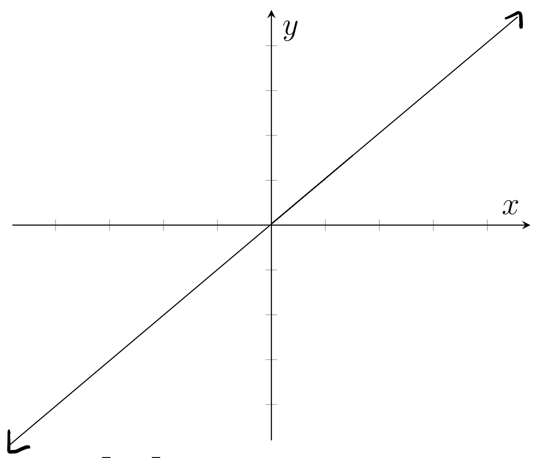
$\|\vec{v}\| = \sqrt{0^2+2^2} = \sqrt{4} = 2.$

2. True or False? ____ Any list of 7 real numbers is a vector in $\mathbb{R}^7$.

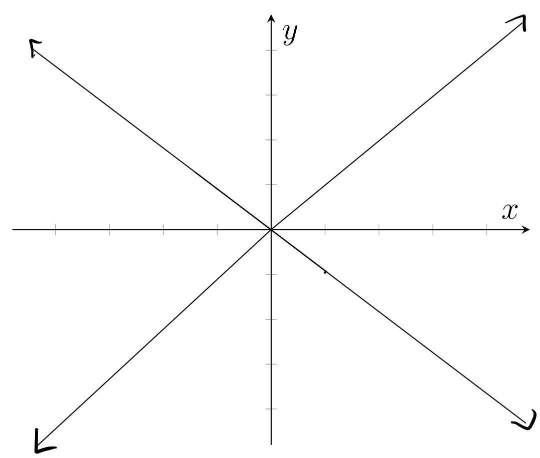
True

3. Linear Combinations:

(a) Let $\boldsymbol{u}$ be the vector $\begin{bmatrix}1\\1\end{bmatrix}$ in $\mathbb{R}^2$. What does the set of all possible combinations, $c\boldsymbol{u}$, $c \in \mathbb{R}$ look like? This is called the **span** of $\boldsymbol{u}$, or span$\{\boldsymbol{u}\}$.
Hint: Try some values for c such as $c = 2, c = -4$, etc.



(b) Let $\boldsymbol{v}$ be another vector $\begin{bmatrix}1\\-1\end{bmatrix}$ in $\mathbb{R}^2$. What does the set of all combinations, $c_1\boldsymbol{u} + c_2\boldsymbol{v}$, where $c_1, c_2 \in \mathbb{R}$, look like? This is the span$\{\boldsymbol{u}, \boldsymbol{v}\}$.
Hint: Try $c_1 = 0$ and $c_2 = 0$ first, then some combinations of the two.



The whole plane $\mathbb{R}^2$

(c) What about if $\boldsymbol{v} = \begin{bmatrix}-1\\-1\end{bmatrix}$? What does the set of all combinations, $c_1\boldsymbol{u} + c_2\boldsymbol{v}$, where $c_1, c_2 \in \mathbb{R}$, look like?

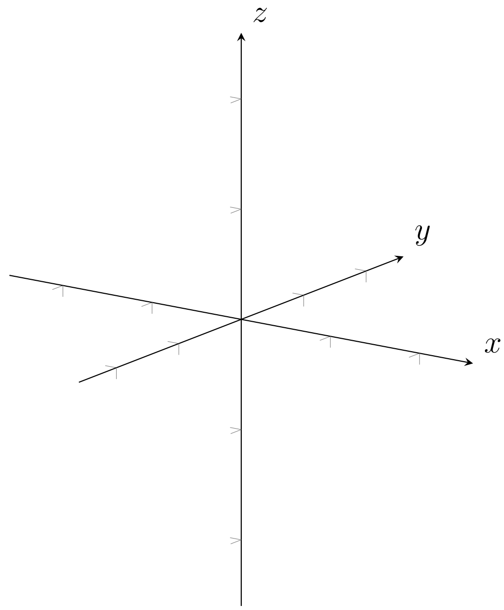
The same line from part (a), since $\vec{u}$ and $\vec{v}$ are LD.

(d) Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be arbitrary vectors in $\mathbb{R}^3$. What does the set of all combinations, $c_1\boldsymbol{u} + c_2\boldsymbol{v} + c_3\boldsymbol{w}$, where $c_1, c_2, c_3 \in \mathbb{R}$, look like? This is the span$\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$.

If all 3 are LD, line in $\mathbb{R}^3$

If 2 are LD, plane in $\mathbb{R}^3$

If all 3 are LI, all of $\mathbb{R}^3$



4. Given the set of vectors:

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \qquad \boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Compute the following:

(a) The unit vector in the direction of $\boldsymbol{u}$.

$\|\vec{u}\| = \sqrt{1^2+3^2+4^2} = \sqrt{1+9+16} = \sqrt{26}$

$$\frac{\vec{u}}{\|\vec{u}\|} = \frac{1}{\sqrt{26}}\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{26} \\ 3/\sqrt{26} \\ 4/\sqrt{26} \end{bmatrix}$$

(b) The unit vector in the direction of $\boldsymbol{v}$.

$\|\vec{v}\| = \sqrt{0^2+1^2+2^2} = \sqrt{1+4} = \sqrt{5}$

$$\frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{5}}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

(c) The dot product, $\boldsymbol{u} \cdot \boldsymbol{v}$.

$\vec{u}\cdot\vec{v} = 1\cdot 0 + 3\cdot 1 + 4\cdot 2 = 0 + 3 + 8 = 11.$

(d) The cosine of the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} = \frac{11}{\sqrt{26}\cdot\sqrt{5}} = \frac{11}{\sqrt{130}}.$$

5. Cosine Formula:

(a) What does the set of all unit vectors in $\mathbb{R}^2$ look like?

A circle with radius 1 centered at the origin.

(b) Consider the unit vector $\boldsymbol{U} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ and the vector along the x-axis, $\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. What is $\boldsymbol{U} \cdot \boldsymbol{i}$?

$\vec{U} \cdot \vec{i} = \cos\alpha \cdot 1 + \sin\alpha \cdot 0 = \cos\alpha.$

(c) Now instead consider a generic unit vector $\boldsymbol{u} = \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix}$. What is $\boldsymbol{u} \cdot \boldsymbol{U}$? Consider the identity $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, let $\theta = \alpha - \beta$, and simplify your answer.

$\vec{u} \cdot \vec{U} = \cos\beta\cos\alpha + \sin\beta\sin\alpha = \cos(\alpha-\beta) = \cos\theta.$

(d) Suppose we have two vectors, u and v that are not unit vectors. Make these vectors unit by dividing by their lengths. Then, apply the formula in (c) to verify the Cosine formula: $\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}$. What is θ here and how does it relate to the α and β in (c)? *Hint:* It might be useful to draw out the vectors.

$\|\vec{u}\| = \sqrt{\cos^2\beta + \sin^2\beta} = 1.$

$\|\vec{U}\| = \sqrt{\cos^2\alpha + \sin^2\alpha} = 1.$

$\vec{u} \cdot \vec{U} = \cos\theta.$

$$\frac{\vec{u}\cdot\vec{U}}{\|\vec{u}\| \cdot \|\vec{U}\|} = \frac{\cos\theta}{1 \cdot 1} = \cos\theta.$$

Sec 1.3:

1. Consider the following vectors and matrices

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \qquad \boldsymbol{u} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \end{bmatrix}$$

Compute the following vector-matrix products.

(a) $A\boldsymbol{u}$

$$A\vec{u} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 1\cdot 4 + 2\cdot(-1) \\ 3\cdot 4 + 0\cdot(-1) \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \end{bmatrix}$$

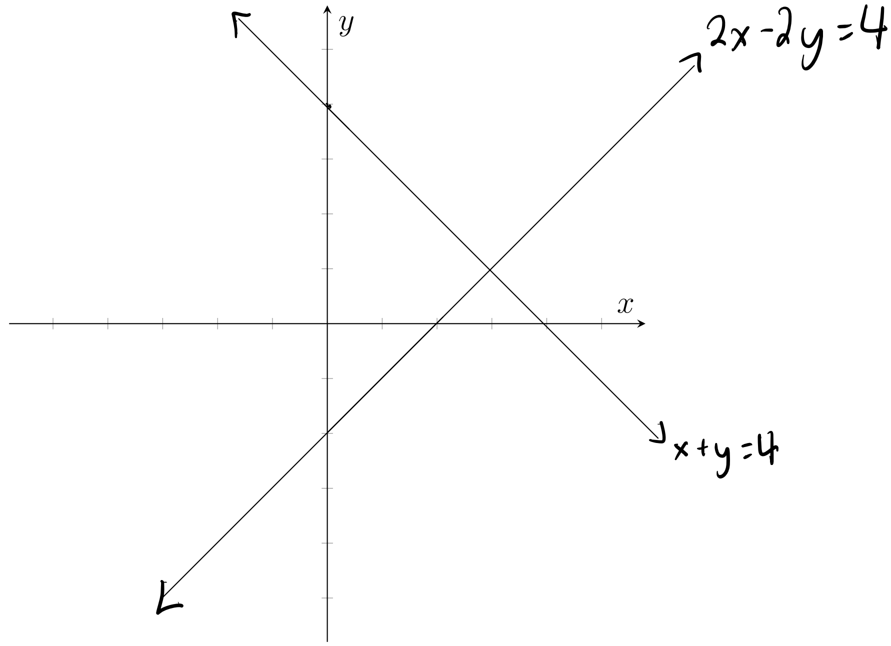
(b) $B\boldsymbol{v}$

$$B\vec{v} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 2\cdot 2 + 3\cdot(-1) \\ 3\cdot 1 + 1\cdot 2 + 0\cdot(-1) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

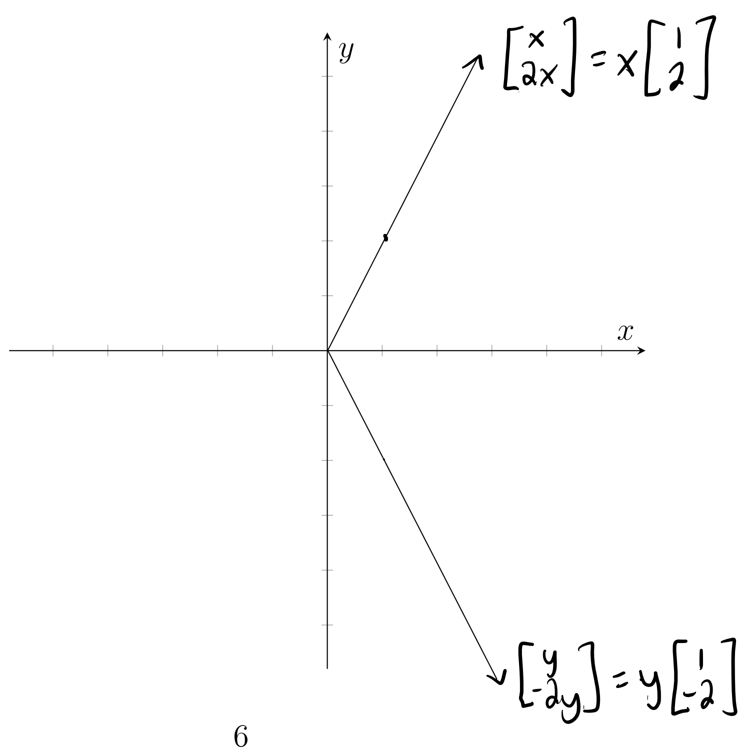
2. Consider the system of equations:

$$x + y = 4 \tag{1a}$$
$$2x - 2y = 4. \tag{1b}$$

(a) Draw the row picture (two intersecting lines).

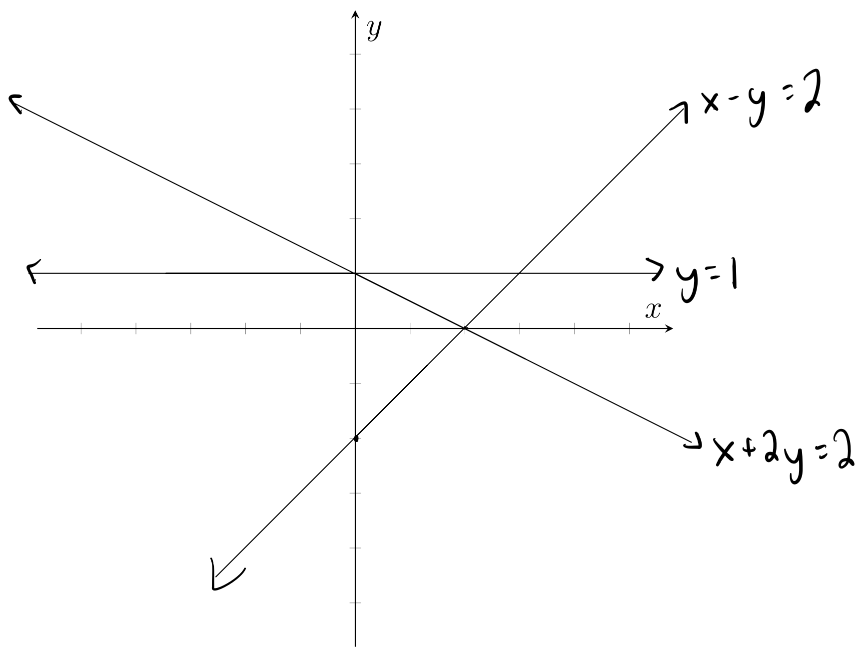


(b) Draw the column picture (combination of two columns equal to the column vector (4,4) on the right side).



3. Sketch these three lines and decide if the equations are solvable:

$$x + 2y = 2 \tag{2a}$$
$$x - y = 2. \tag{2b}$$
$$y = 1. \tag{2c}$$



Not solvable; no common intersection

4. Find A^{-1} by rewriting the following matrix-vector system

$$A\boldsymbol{x} = \boldsymbol{b} \qquad \Longrightarrow \qquad \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}.$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 + 2x_3 \\ x_1 - x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_2 + 2x_3 = b_1 \Rightarrow x_2 = b_1 - 2b_3 \\ x_1 - x_2 = b_2 \Rightarrow x_1 = b_1 + b_2 - 2b_3 \\ x_3 = b_3 \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 + b_2 - 2b_3 \\ b_1 - 2b_3 \\ b_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{A^{-1}}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

5. Write the following system of equations as a matrix-vector system.

Hint: Write $\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ so that $A\boldsymbol{x} = \boldsymbol{b}$.

$$\begin{aligned} 2x + 2y &= 9 \\ -y + z &= 1 \\ x + 6z &= 0 \end{aligned}$$

$$\begin{bmatrix} 2 & 2 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}$$

6. Write the following matrix-vector system as a system of linear equations

$$\begin{bmatrix} 1 & 5 & 2 \\ 3 & 0 & -1 \\ 8 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + 5x_2 + 2x_3 \\ 3x_1 - x_3 \\ 8x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 9 \end{bmatrix}$$

$$\begin{cases} x_1 + 5x_2 + 2x_3 = -2 \\ 3x_1 - x_3 = 7 \\ 8x_1 + 2x_2 = 9 \end{cases}$$