Midterm Review 3/10

## Invertibility

An $n\times n$ matrix $A$ is invertible if $\exists A^{-1}$ s.t. $AA^{-1}=A^{-1}A=I$

### Fall 2017 Midterm Q3

$$\begin{bmatrix} 1 & 1 & 1 \\ \textcircled{1} & 2 & t \\ \textcircled{1} & 4 & t^2 \end{bmatrix} \begin{matrix} \\ R_2-R_1 \rightarrow \\ R_3-R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & t-1 \\ 0 & \textcircled{3} & t^2-1 \end{bmatrix} \begin{matrix} \\ \rightarrow \\ R_3-3R_2 \end{matrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & t-1 \\ 0 & 0 & \boxed{t^2-3t+2} \end{bmatrix}$$

$t^2-3t+2=0$

$(t-1)(t-2)=0$

$t=1,\ t=2$

### Quiz 3 Q3

$$\begin{bmatrix} 1 & 3 & a \\ \textcircled{3} & 1 & a \\ \textcircled{a} & a & a \end{bmatrix} \begin{matrix} \\ R_2-3R_1 \rightarrow \\ R_3-aR_1 \end{matrix} \begin{bmatrix} 1 & 3 & a \\ 0 & -8 & -2a \\ 0 & \textcircled{-2a} & a-a^2 \end{bmatrix} \begin{matrix} \\ \rightarrow \\ R_3-\frac{1}{4}aR_2 \end{matrix} \begin{bmatrix} 1 & 3 & a \\ 0 & -8 & -2a \\ 0 & 0 & a-\frac{1}{2}a^2 \end{bmatrix}$$

$a-\frac{1}{2}a^2 \neq 0$

$a(1-\frac{1}{2}a)\neq 0$

$\boxed{a\neq 0}$

$1-\frac{1}{2}a\neq 0 \Rightarrow \frac{1}{2}a\neq 1 \Rightarrow \boxed{a\neq 2}$

### Quiz 2 Q10

$$P=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad P'=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$[*\ \ *\ \ *\ \ *]$

$$\begin{tikzcd}[column sep=small] \overset{P}{(4} \arrow[rrr, bend right, leftrightarrow] & \times 4) \arrow[r, bend right, leftrightarrow] & \times (n & \overset{A}{\times m)} \end{tikzcd}$$

PA is $(4\times m)$

$\underbrace{P\underbrace{P'A}}$

### Quiz 3 Q1.1, 1.2

1.1) If $MN=N$ then $M=I$ where $I$ is the identity matrix?

False - need $M$ to be arbitrary.

$\forall A,\ AI=IA=A$

1.2) If $M$ and $N$ are invertible, then $MN=NM$.

False

### Rec 4 Q2, Q3 (3.1)

2) $W=\{\vec{x}\in\mathbb{R}^3 : \overset{\downarrow}{x_1-x_2+2x_3=0},\ \overset{\downarrow}{3x_2-x_3=0}\}$

$0-0+2\cdot0=0,\ 3\cdot0-0=0$, so $\vec{0}\in W$

Let $\vec{x},\vec{y}\in W$, $\alpha,\beta\in\mathbb{R}$.

$$\alpha\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}+\beta\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} \alpha x_1+\beta y_1 \\ \alpha x_2+\beta y_2 \\ \alpha x_3+\beta y_3 \end{bmatrix}$$

$$\begin{aligned}(\alpha x_1+\beta y_1)-(\alpha x_2+\beta y_2)+2(\alpha x_3+\beta y_3) &= \alpha x_1-\alpha x_2+2\alpha x_3+\beta y_1-\beta y_2+2\beta y_3 \\ &= \alpha(\underset{\overset{\|}{0}}{x_1-x_2+2x_3})+\beta(\underset{\overset{\|}{0}}{y_1-y_2+2y_3}) \\ &= 0\end{aligned}$$

$\Rightarrow \alpha\vec{x}+\beta\vec{y}\in W$

$\therefore W$ is a subspace.

3) Let $U=\{$nonsingular $2\times 2$ matrices$\}$

nonsingular $\Leftrightarrow$ invertible.

1) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin U$

2) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin U$

3) $0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin U$

3b) Let $V = \{\text{singular } 2\times 2 \text{ matrices}\}$.

singular $\Leftrightarrow$ not invertible

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin V$$

## Quiz 4 Q5

$F = \{(x,y,z) \in \mathbb{R}^3 : x^2 - z^2 = 0\}$

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \notin F$$

$3^2 - 1^2 = 9 - 1 = 8$

## Quiz 4 Q1.5

$$\begin{bmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \qquad x_4\begin{bmatrix} \\ \\ \\ \end{bmatrix} + x_5\begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$\underbrace{\quad\quad\quad}_{\text{pivots}} \underbrace{\quad\quad}_{\text{free}}$

## LDU Factorization

$$A = \boxed{L}U = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}}_{\text{leave alone}} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -5 \end{bmatrix}}_{} \begin{matrix} \cdot\frac{1}{1} \leftarrow \\ \cdot\frac{1}{2} \leftarrow \\ \cdot -\frac{1}{5} \leftarrow \end{matrix}$$

$$\boxed{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{bmatrix} \qquad \boxed{U'} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

## Spring 2017 Midterm A Q2

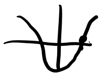
$\vec{u} = (1,3,1),\ \vec{v} = (2,1,-3)$

$\|\vec{u}\| = \sqrt{1^2+3^2+1^2} = \sqrt{1+9+1} = \sqrt{11}$

$\|\vec{v}\| = \sqrt{2^2+1^2+(-3)^2} = \sqrt{4+1+9} = \sqrt{14}$

$\vec{u}\cdot\vec{v} = 1\cdot 2 + 3\cdot 1 + 1\cdot(-3) = 2+3-3 = 2$

$\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} = \dfrac{2}{\sqrt{11}\cdot\sqrt{14}} = \dfrac{2}{\sqrt{154}}$



## Quiz 2, Q4

$\exists(A+B) \Rightarrow \exists(AB^T)$

$\exists(A+B) \Rightarrow$ A and B are both $m\times n$

$\Rightarrow B^T$ is $n\times m$

$AB^T$ is $(m\times n)\times(n\times m)$ ✓

## HW 1 Exercise V

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a LD set in $\mathbb{R}^n$, and $\vec{v}_4 \in \mathbb{R}^n$.

Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is LD, $\exists c_1, c_2, c_3$, not all zero, s.t. $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = 0$.

$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + 0\vec{v}_4 = 0$

## HW 2, Q3.2

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} \rightarrow U = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

$$E_{*1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, E_{*2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$E_{43}E_{*2}E_{*1}A = U$

$$L = (E_{43}E_{*2}E_{*1})^{-1} = E_{*1}^{-1}E_{*2}^{-1}E_{43}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

$a \neq 0$

$b-a \neq 0 \Rightarrow a \neq b$

$c-b \neq 0 \Rightarrow b \neq c$

$d-c \neq 0 \Rightarrow c \neq d$

## Fall 2017 Midterm 1, Q7

$\mathbb{P}_2$ is the space of polynomials w/ deg $\leq 2$

Basis $\{1, x, x^2\}$

$$\boxed{f(x) = \underset{\uparrow_{1}}{a_0} + \underset{\uparrow_{x}}{a_1} x + \underset{\uparrow_{x^2}}{a_2} x^2}$$

$\{(x-1)^2, x-1, 1\}$

## Spring 2017 Midterm 1, MC 9

$\{1, x, x^2\}$

## HW 1, QI.4

T or F? If $\vec{u}$ and $\vec{v}$ are perpendicular unit vectors, then $\|\vec{u}+3\vec{v}\| = \sqrt{10}$.

$\vec{u} \cdot \vec{v} = 0$, $\|\vec{u}\| = \|\vec{v}\| = 1$.

$$\begin{aligned} \|\vec{u}+3\vec{v}\|^2 &= (\vec{u}+3\vec{v}) \cdot (\vec{u}+3\vec{v}) = \vec{u}\cdot\vec{u} + \cancel{3\vec{u}\cdot\vec{v}} + \cancel{3\vec{u}\cdot\vec{v}} + 9\vec{v}\cdot\vec{v} \\ &= \|\vec{u}\|^2 + 9\|\vec{v}\|^2 \\ &= 1^2 + 9 \cdot 1^2 \\ &= 10 \end{aligned}$$

$\|\vec{u}+3\vec{v}\| = \sqrt{10}$

True.

## HW 3, Q1.a

Let $\vec{a} = (a_1, a_2) \in V$, $c \in \mathbb{R}$

$c\vec{a} = c(a_1, a_2) = (0, a_2) \neq \vec{a}$, $\forall c \in \mathbb{R}$

So there's no identity.

## Spring 2017 Midterm 1 Q6

A & B invertible.

A+B?

Let $A = I$, $B = -I$

$$A+B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A+B not necessarily invertible.

HW 3 Q1.2