# Chapter 3: The Four Fundamental Subspaces

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## 1 Vector Spaces

**Definition 1.1.** *A* ***vector space*** $V$ *defined over* $\mathbb{R}$ *consists of a set on which addition and scalar multiplication are defined so that for each pair of elements* $v, w \in V$, *there is a unique element* $v + 2 \in V$, *and for each element* $c \in \mathbb{R}$ *and* $v \in V$, *there is a unique element* $cv \in V$, *such that the following conditions hold:*

1. *For all* $v, w \in V$, $v + w = w + v$ *(commutativity).*
2. *For all* $u, v, w \in V$, $(u + v) + w = u + (v + w)$ *(associativity under addition).*
3. *There exists an element* $0 \in V$ *such that* $v + 0 = v$ *for each* $v \in V$ *(identity element under vector addition).*
4. *For each element* $v \in V$, *there exists an element* $-v \in V$ *such that* $v + (-v) = 0$ *(additive inverse).*
5. *There exists an element* $1 \in \mathbb{R}$ *such that* $1v = v$ *for each* $v \in V$ *(identity element under scalar multiplication).*
6. *For each pair of elements* $c, d \in \mathbb{R}$, *and each* $v \in V$, $(cd)v = c(dv)$ *(associativity under scalar multiplication).*
7. *For each element* $c \in \mathbb{R}$, *and each pair* $v, w \in V$, $c(v + w) = cv + cw$ *(distributivity of scalar multiplication over vector addition).*
8. *For each pair of elements* $c, d \in \mathbb{R}$, *and each* $v \in V$, $(c + d)v = cv + dv$ *(distributivity of scalar addition over scalar multiplication).*

## 2 Subspaces

**Definition 2.1.** *Let* $V$ *be a vector space. A subset* $S \subseteq V$ *is a* ***subspace*** *of* $V$ *if the following hold:*

1. $\vec{0} \in S$.
2. *If* $\vec{x}, \vec{y} \in S$, *then* $\vec{x} + \vec{y} \in S$.
3. *If* $\vec{x} \in S$ *and* $c$ *is a scalar, then* $c\vec{x} \in S$.

## 3 Column Space

**Definition 3.1.** *The* ***column space*** *of a matrix consists of all linear combinations of its columns. So if* $A = \begin{bmatrix} \vec{a_1} & \vec{a_2} & \cdots & \vec{a_n} \end{bmatrix}$ *is an* $m \times n$ *matrix, where* $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n} \in \mathbb{R}^m$, *then* $ColA = span\{\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n}\}$. $ColA$ *is a subspace of* $\mathbb{R}^m$.

**Proposition 3.2.** *The system* $A\vec{x} = \vec{b}$ *is solvable if and only if* $\vec{b} \in C(A)$.

## 4 Nullspace

**Definition 4.1.** *The* ***nullspace*** *of a matrix* $A$ *consists of all solutions* $\vec{x}$ *to the system* $A\vec{x} = \vec{0}$. *So if* $A = \begin{bmatrix} \vec{a_1} & \vec{a_2} & \cdots & \vec{a_n} \end{bmatrix}$ *is an* $m \times n$ *matrix, where* $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n} \in \mathbb{R}^m$, *then* $NulA = \{\vec{x} : A\vec{x} = \vec{0}\}$. $NulA$ *is a subspace of* $\mathbb{R}^n$.

## 5 Row Echelon Form (REF)

**Definition 5.1.** *An $m \times n$ matrix is in **row echelon form (REF)** if:*

1. *All rows consisting entirely of zeros lie beneath all nonzero rows.*
2. *The first nonzero element in any row, called a **pivot,** must lie to the right of any pivot above it.*

**Algorithm 5.2.** ***(REF)*** *To find the REF of a matrix $A$, find the pivots and use them to make all elements below them equal zero.*

**Definition 5.3.** *A **pivot column** in a matrix in REF is a column that contains exactly one pivot. A **free column** in a matrix in REF is a column that contains no pivots.*

## 6 Reduced Row Echelon Form (RREF)

**Algorithm 6.1.** ***(RREF)*** *To find the RREF of a matrix $A$ ::*

1. *Find the REF of $A$ using Algorithm 5.2.*
2. *Use the obtained pivots to make all elements above them equal zero.*
3. *Finally, make all pivots equal* 1.

**Proposition 6.2.** *The RREF of a matrix $A$ has the same nullspace as the original matrix $A$.*

**Algorithm 6.3.** ***(Nullspace)*** *To find the nullspace of a matrix $A$,*

1. *Find the RREF of $A$.*
2. *Use the RREF and back substitution to solve the system $A\vec{x} = \vec{0}$.*
3. *NulA is the set of solutions $\vec{x}$.*

## 7 Rank

**Definition 7.1.** *The **rank** $r$ of an $m \times n$ matrix $A$ is the number of pivots in its REF.*

## 8 Complete Solutions

**Theorem 8.1.** *Let $A$ be an $m \times n$ matrix such that $m < n$. In this case, we are guaranteed to have free columns, and the system $A\vec{x} = \vec{b}$ will have more unknowns than equations, so it will have free variables associated with the free columns. Thus, the system $A\vec{x} = \vec{b}$ will always have either an infinite number of solutions or no solutions.*

**Definition 8.2.** *The **particular solution** $\vec{x_p}$ of a system is obtained by setting the free variables to zero. $\vec{x_p}$ solves $A\vec{x_p} = \vec{b}$.*

**Definition 8.3.** *The **nullspace solution** $\vec{x_n}$ of a system is obtained by setting $\vec{b}$ to $\vec{0}$. There are $n - r$ nullspace solutions which solve $A\vec{x_n} = \vec{0}$, where $r$ is the rank of $A$.*

**Definition 8.4.** *The **complete solution** to $A\vec{x} = \vec{b}$ can be expressed as $\vec{x} = \vec{x_p} + \vec{x_n}$.*

## 9 Ranks and Systems

**Proposition 9.1.** *Let $A$ be an $m \times n$ matrix with rank $r$.*

1. *The $r$ pivot columns of $A$ are linearly independent.*
2. *$A$ has $n - r$ free columns.*
3. *Since ColA is the span of the pivot columns of $A$, the column space spans an $r$-dimensional space.*
4. *The dimension of NulA is $n - r$.*

**Definition 9.2.** *Let $A$ be an $m \times n$ matrix. Then $A$ has **full column rank** $r = n$ if:*

1. *All columns of A are pivot columns.*
2. *All columns of A are linearly independent.*
3. *There are no free columns, which implies that there are no free solutions.*
4. $NulA = \{\vec{0}\}$.
5. *If* $A\vec{x} = \vec{b}$ *has a solution, then it has exactly one solution.*

**Definition 9.3.** *Let A be an* $m \times n$ *matrix. Then A has* ***full row rank*** $r = m$ *if:*

1. *All rows of A have pivot positions.*
2. *All rows of A are linearly independent.*
3. *There are* $n - r = n - m$ *nullspace solutions.*
4. *ColA spans all of* $\mathbb{R}^m$.
5. $A\vec{x} = \vec{b}$ *has a solution for every* $\vec{b}$.

**Proposition 9.4.** ***(Rank and Solvability)*** *Let A be an* $m \times n$ *matrix with rank r. The solutions to* $A\vec{x} = \vec{b}$ *can be classified as follows:*

1. *If* $r = m = n$, *then A is a square invertible matrix, so* $A\vec{x} = \vec{b}$ *has exactly one solution.*
2. *If* $r = m, r < n$, *then A is short and wide, so* $A\vec{x} = \vec{b}$ *has an infinite number of solutions.*
3. *If* $r < m, r = n$, *then A is tall and thin, so* $A\vec{x} = \vec{b}$ *has either no solution or exactly one solution.*
4. *If* $r < m, r < n$, *then A does not have full rank, so* $A\vec{x} = \vec{b}$ *has either no solutions or an infinite number of solutions.*

## 10 Row Space

**Definition 10.1.** *The* ***row space*** *RowA of an* $m \times n$ *matrix A is the span of the nonzero rows of its REF.*

**Theorem 10.2.** *Let A be an* $m \times n$ *matrix. Then RowA =*$ColA^T$ *=span{linearly independent columns of* $A^T$*}.*

## 11 Basis of a Vector Space

**Definition 11.1.** *A* ***basis*** $\beta$ *for a vector space V is a set of vectors* $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ *such that*

1. $vecv_1, \vec{v_2}, \ldots, \vec{v_n}$ *are linearly independent.*
2. $vecv_1, \vec{v_2}, \ldots, \vec{v_n}$ *span V.*

**Definition 11.2.** *The* ***standard basis*** *for* $\mathbb{R}^n$ *is* $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$.

**Proposition 11.3.** *The pivot columns of a matrix A form a basis for ColA.*

**Proposition 11.4.** *The nullspace solutions of a matrix A form a basis for NulA.*

**Theorem 11.5.** *If V is a vector space and* $\vec{v} \in V$, *then there is a unique way to write* $\vec{v}$ *as a linear combination of the basis vectors of V.*

## 12 Dimension of a Vector Space

**Definition 12.1.** *The* ***dimension*** *of a vector space V is the number of vectors in a basis* $\beta$ *for V.*

**Proposition 12.2.** *For an* $m \times n$ *matrix A with rank r,*

1. $dim(ColA) = r$.
2. $dim(RowA) = r$.
3. $dim(NulA) = n - r$.

**Theorem 12.3.** *If* $\vec{v_1}, \ldots, \vec{v_m}$ *and* $\vec{w_1}, \ldots, \vec{w_n}$ *are both bases for a vector space V, then* $m = n$.

# 13 Matrix Subspaces

**Definition 13.1.** *Let* $A = \begin{bmatrix} \vec{a_1} & \vec{a_2} & \cdots & \vec{a_n} \end{bmatrix}$ *be an* $m \times n$ *matrix, where* $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n} \in \mathbb{R}^m$. *Then the* ***four fundamental subspaces*** *associated with* $A$ *are:*

1. *The column space ColA =span{pivot columns}* $\subseteq \mathbb{R}^m$.
2. *The row space RowA =ColA*$^T \subseteq \mathbb{R}^n$.
3. *The nullspace NulA* $= \{\vec{x} : A\vec{x} = \vec{0}\} \subseteq \mathbb{R}^n$.
4. *The left nullspace NulA*$^T = \{\vec{y} : A^T\vec{y} = \vec{0}\} \subseteq \mathbb{R}^m$.

**Proposition 13.2.** *If* $A$ *is an* $m \times n$ *matrix with rank* $r$, *then*

1. *dim(ColA)=r.*
2. *dim(RowA)=r.*
3. *dim(NulA)=n-r.*
4. *dim(NulA*$^T$*)=m-r.*