# Chapter 1: Intro to Vectors

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## 1 Vectors

**Definition 1.1.** *A (column)* ***vector*** *is a list of real numbers arranged in a column. We write vectors with arrows over them, as in*

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

*where $x_1, x_2, \ldots, x_n \in \mathbb{R}$ are real numbers called the* ***components*** *of $\vec{x}$. An* ***n-dimensional vector*** *is a vector with $n$ components.*

**Definition 1.2.** *The set of all $n$-dimensional vectors is* ***n-dimensional Euclidean space,*** *denoted $\mathbb{R}^n$.*

**Definition 1.3.** *(**Vector addition**) For any $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$, define*

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}.$$

**Definition 1.4.** *(**Scalar multiplication**) For any $x_1, \ldots, x_n, c \in \mathbb{R}$, define*

$$c \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}.$$

**Definition 1.5.** *Let $\vec{0}$ denote the* ***zero vector,*** *whose components are all zero:*

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

*where the number of components is in context. For example, if $\vec{0} \in \mathbb{R}^2$, then $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.*

**Definition 1.6.** *Let $(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n})$ be a list of vectors in $\mathbb{R}^m$. A* ***linear combination*** *of $(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n})$ is a vector of the form*

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n}$$

*for some scalars $c_1, c_2, \ldots, c_n \in \mathbb{R}$, which are called the* ***weights*** *or* ***coefficients****.*

## 2 Lengths and Dot Products

**Definition 2.1.** *Let $\vec{x}, \vec{y} \in \mathbb{R}^n$, where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$. The* ***dot product*** *$\vec{x} \cdot \vec{y}$ of $\vec{x}$ and $\vec{y}$ is given by*

$$\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

**Proposition 2.2.** *For all $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$ and scalars $c$,*

1. $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$ *(commutative).*
2. $(\vec{x} + \vec{y}) \cdot \vec{z} = \vec{x} \cdot \vec{z} + \vec{y} \cdot \vec{z}$ *(distributive over addition).*
3. $(c\vec{x}) \cdot \vec{y} = c(\vec{x} \cdot \vec{y})$.
4. $\vec{x} \cdot \vec{x} \geq 0$.
5. $\vec{x} \cdot \vec{x} = 0$ *if and only if* $\vec{x} = \vec{0}$.

**Definition 2.3.** *The **length** or **norm** of a vector $\vec{x} \in \mathbb{R}^n$ is given by*

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

*or equivalently,* $\|\vec{x}\|^2 = \vec{x} \cdot \vec{x}$.

**Proposition 2.4.** *For all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and scalars $c$,*

1. $\|\vec{x}\| \geq 0$.
2. $\|\vec{x}\| = 0$ *if and only if* $\vec{x} = \vec{0}$.
3. $\|c\vec{x}\| = |c| \cdot \|\vec{x}\|$.
4. $\|\vec{x} + \vec{y}\|^2 + \|\vec{x} - \vec{y}\|^2 = 2(\|\vec{x}\|^2 + \|\vec{y}\|^2)$ *(parallelogram law).*
5. $\vec{x} \cdot \vec{y} = \frac{1}{4}(\|\vec{x} + \vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2)$.

**Definition 2.5.** *The **distance** between two points $x, y \in \mathbb{R}^n$ is*

$$dist(x, y) = \|y - x\|.$$

*This is just the length of the vector from $x$ to $y$.*

**Definition 2.6.** *A vector $\vec{x} \in \mathbb{R}^n$ is a **unit vector** if $\|\vec{x}\| = 1$.*

**Definition 2.7.** *Let $\vec{x} \in \mathbb{R}^n$ be a nonzero vector. The **unit vector in the direction of** $\mathbf{x}$ is the vector $\frac{\vec{x}}{\|\vec{x}\|}$.*

**Definition 2.8.** *Two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ are **orthogonal** or **perpendicular** if $\vec{x} \cdot \vec{y} = 0$.*

**Proposition 2.9.** ***(Cosine Formula)** The **angle** $\theta$ between two nonzero vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ is given by:*

$$\frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|} = \cos\theta.$$

**Theorem 2.10.** ***(Cauchy-Schwarz Inequality)** For all $\vec{x}, \vec{y} \in \mathbb{R}^n$,*

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|.$$

**Proposition 2.11.** ***(Triangle Inequality)** For all $\vec{x}, \vec{y} \in \mathbb{R}^n$,*

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|.$$

# 3 Spans, Linear Dependence, and Linear Independence

**Definition 3.1.** *Let $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}$ be a set of vectors in $\mathbb{R}^n$. The **span** of $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}$ is the set of all linear combinations of $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}$, and is denoted span$\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}\}$.*
*In symbols:* span$\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}\} = \{c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_m\vec{v_m} : c_1, c_2, \ldots, c_m \in \mathbb{R}\}$.

**Definition 3.2.** *Suppose $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m} \in \mathbb{R}^n$ and consider the equation*

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_m\vec{v_m} = \vec{0}.$$

*This equation always has the solution $c_1 = c_2 = \cdots = c_m = 0$, called the **trivial solution**.*

1. *If $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_m\vec{v_m} = \vec{0}$ has only the trivial solution, then $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}\}$ is **linearly independent**.*
2. *If $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_m\vec{v_m} = \vec{0}$ has nontrivial solutions, then $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}\}$ is **linearly dependent**.*

**Proposition 3.3.** *A set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}\}$ of vectors in $\mathbb{R}^n$ is **linearly dependent** if and only if one of the vectors is in the span of the others.*

# 4 Matrices

**Definition 4.1.** *A **matrix** is a rectangular array of numbers. We say a matrix is $m \times n$, i.e. "$m$ by $n$" if it has $m$ rows and $n$ columns. If $A$ is $m \times n$ with entries $a_{ij}$, then we write*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

*A matrix is called **square** if it is $n \times n$.*

**Definition 4.2.** *The entries $a_{11}, a_{22}, a_{33}, \ldots$ are the **diagonal entries**, which form the **main diagonal** of the matrix. A **diagonal matrix** is a square matrix whose non-diagonal entries are all zeros.*

**Definition 4.3.** *The $n \times n$ **identity matrix** $I_n$ is the diagonal matrix with all diagonal entries equal to 1. $I_n$ is special because $I_n\vec{v} = \vec{v}$ for all $\vec{v} \in \mathbb{R}^n$.*

**Definition 4.4.** *The $m \times n$ **zero matrix** is the $m \times n$ matrix 0 with all zero entries.*

**Definition 4.5.** *The **transpose** of an $m \times n$ matrix $A$ is the $n \times m$ matrix $A^T$ whose rows are the columns of $A$. In other words, the $ij$-th entry of $A^T$ is $a_{ji}$.*

**Proposition 4.6.** *A linear combination of $n$ vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n} \in \mathbb{R}^m$ can be expressed as an $m \times n$ matrix $A = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \end{bmatrix}$ multiplying a vector $\vec{x} \in \mathbb{R}^n$:*

$$\begin{bmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{v_1} + x_2\vec{v_2} + \cdots + x_n\vec{v_n}.$$