Chp 1: Intro to Vectors

1.1 Vectors and Linear Combinations

Vector addition: $\vec{v}=\begin{bmatrix}v_1\\v_2\end{bmatrix}$ and $\vec{w}=\begin{bmatrix}w_1\\w_2\end{bmatrix}$ add to $\vec{v}+\vec{w}=\begin{bmatrix}v_1+w_1\\v_2+w_2\end{bmatrix}$.

Scalar multiplication: If $\vec{v}=\begin{bmatrix}v_1\\v_2\end{bmatrix}$ and c is a scalar, then $c\vec{v}=\begin{bmatrix}cv_1\\cv_2\end{bmatrix}$.

The sum of $c\vec{v}$ and $d\vec{w}$ is a linear combination (LC) $c\vec{v}+d\vec{w}$. The LCs $c\begin{bmatrix}v_1\\v_2\end{bmatrix}+d\begin{bmatrix}w_1\\w_2\end{bmatrix}$ fill the xy-plane. The LCs $c\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}+d\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}$ fills a plane in xyz space.

The length of $\vec{v}=\begin{bmatrix}v_1\\v_2\end{bmatrix}$ is $\|\vec{v}\|=\sqrt{v_1^2+v_2^2}$.

The dot product of $\vec{v}$ and $\vec{w}$ is $\vec{v}\cdot\vec{w}=v_1w_1+v_2w_2$.

The column space of a matrix A is all combinations $A\vec{x}$ for all $\vec{x}$.

If a matrix A has n column vectors $v_1,\dots,v_n$ in m-dimensional space, then A is the $m\times n$ matrix $A=[v_1\ v_2\ \cdots\ v_n]$.

To solve an equation $c\begin{bmatrix}v_1\\v_2\end{bmatrix}+d\begin{bmatrix}w_1\\w_2\end{bmatrix}=\begin{bmatrix}b_1\\b_2\end{bmatrix}$, use elimination to solve $\begin{cases}cv_1+dw_1=b_1\\cv_2+dw_2=b_2\end{cases}$. In matrix form, this is $\begin{bmatrix}v_1&w_1\\v_2&w_2\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix}=\begin{bmatrix}b_1\\b_2\end{bmatrix}$.

If $\vec{v}$, $\vec{w}$, and $\vec{0}$ aren't on the same line, then there's exactly one solution c,d. Then the LCs of $\vec{v}$ and $\vec{w}$ exactly fill the xy-plane. In this case, $\vec{v}$ and $\vec{w}$ are linearly independent (LI), and the 2×2 matrix $A=[\vec{v}\ \vec{w}]$ is invertible.

If $\vec{v}$ and $\vec{w}$ lie on the same line through $(0,0)$, and $\vec{b}=\begin{bmatrix}b_1\\b_2\end{bmatrix}$ isn't on that line, then the system has no solution. Then the LCs of $\vec{v}$ and $\vec{w}$ fill the line they're on, but not the plane. In this case, $\vec{v}$ and $\vec{w}$ are linearly dependent (LD)

LCs of two LI vectors $\vec{v}$ and $\vec{w}$ in 2-dim space can produce any vector $\vec{b}$ in that plane. So $c\vec{v}+d\vec{w}=\vec{b}$ has a solution.

LCs of two LI vectors $\vec{v}$ and $\vec{w}$ in 3-dim space do not fill the whole 3D space; they only fill a 2D plane.

LCs of three LI vectors in 3D space fill the whole 3D space.

The standard unit basis vectors in 3D space are $\vec{i}=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{j}=\begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{k}=\begin{bmatrix}0\\0\\1\end{bmatrix}$. Any vector $\vec{v}\in\mathbb{R}^3$ is a LC of $\vec{i},\vec{j}$, and $\vec{k}$: $\vec{v}=\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=v_1\vec{i}+v_2\vec{j}+v_3\vec{k}$.

The 3×3 identity matrix is $I=[\vec{i}\ \vec{j}\ \vec{k}]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$. Multiplying by I leaves every vector unchanged.

1.2 Lengths and Angles from Dot Products

The dot product of $\vec{v},\vec{w}\in\mathbb{R}^n$ is the scalar $\vec{v}\cdot\vec{w}=v_1w_1+\dots+v_nw_n$.

A unit vector $\vec{u}$ has length $\|\vec{u}\|=1$. If $\vec{v}\neq0$, then $\frac{\vec{v}}{\|\vec{v}\|}$ is the unit vector in the direction of $\vec{v}$.

$\vec{v}$ and $\vec{w}$ are perpendicular if $\vec{v}\cdot\vec{w}=0$. In this case, $\|\vec{v}+\vec{w}\|^2=\|\vec{v}\|^2+\|\vec{w}\|^2=\|\vec{v}-\vec{w}\|^2$.

Cosine formula: If $\|\vec{v}\|=1$ and $\|\vec{u}\|=1$, then the angle θ b/w $\vec{v}$ and $\vec{u}$ has $\cos\theta=\vec{v}\cdot\vec{u}$.

Cosine formula: If $\vec{v}$ & $\vec{w}$ are nonzero vectors, then the angle θ b/w them has $\cos\theta=\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}$.

Schwarz Inequality: $|\vec{v}\cdot\vec{w}|\leq\|\vec{v}\|\|\vec{w}\|$.

Triangle Inequality: $\|\vec{v}+\vec{w}\|\leq\|\vec{v}\|+\|\vec{w}\|$.

If $\vec{n}\in\mathbb{R}^3$ is a unit vector, then the vectors $\vec{w}\in\mathbb{R}^3$ s.t. $\vec{w}\cdot\vec{n}=0$ fill a 2D plane in $\mathbb{R}^3$.

1.3 Matrices and their Column Spaces

The row picture of $A\vec{x}$ comes from dot products of $\vec{x}$ w/ the rows of A.

The column picture of $A\vec{x}$ comes from a LC of the columns of A: $A\vec{x}=x_1(\text{column } a_1)+\dots+x_n(\text{column } a_n)$.

The columns of a matrix A are independent if the only combination of columns s.t. $A\vec{x}=\vec{0}$ is $\vec{x}=\vec{0}$. Otherwise, the columns are dependent.

The column space $C(A)$ contains all vectors $A\vec{x}$: all combos of the columns.

The span of a set of vectors describes all the LCs of those vectors. So the span of the columns of A is the column space.

The possibilities for column spaces of a 3×3 matrix A are: (1) the whole space $\mathbb{R}^3$, if A has 3 indep columns; (2) a plane in $\mathbb{R}^3$ going through $(0,0,0)$, if A has 2 indep columns; a line in $\mathbb{R}^3$ going through $(0,0,0)$, if A has 1 indep column; or (4) the point $(0,0,0)$, if A is a matrix of zeros.

Every matrix of rank r is the sum of r matrices of rank 1. Rank is the # of indep columns.

A matrix has rank 1 if all the rows are multiples of one row. When the column space is a single line in mD space, the row space is a single line in nD space.

For all matrices, the row rank equals the column rank.

1.4 Matrix Multiplication AB and CR

To multiply matrices A and $B=[b_1\ \cdots\ b_p]$, we take $AB=[Ab_1\ \cdots\ Ab_p]$. The row length of A must equal the column length of B. If A is $m\times n$ and B is $n\times p$, then AB is $m\times p$.

Matrix multiplication is not commutative. In general, $AB\neq BA$.

Matrix multiplication is associative. If A is $m\times n$, B is $n\times p$, and C is $p\times q$, then $(AB)C=A(BC)$ is $m\times q$. It's also distributive: $A(B+C)=AB+AC$.

If A is a a rank one matrix, then all columns of A lie on the same line, and that line is the column space of A. The rows also lie on a line, and that line is the row space of A.

If the column space of A is a line, then the row space of A is also a line.

If $\text{rank}(A)=1$ for an $m\times n$ matrix A, then we can factor A into $A=CR$ where C is $m\times1$ and R is $1\times n$.

If $\text{rank}(A)=r$ for an $m\times n$ matrix A, we can still factor $A=CR$. We go from left to right: if column 1 of A is not all zero, put it into C. Then if column 2 isn't a multiple of column 1, put it into C. Then if column 3 isn't a LC of columns 1 and 2, put it into C. Continue. At the end, C will have $r=\text{rank}(A)=\text{rank}(C)$ columns taken from A. While the n columns of A might be dependent, the r columns of C will be independent. This means that no column of C is a LC of previous columns, and no combination of columns gives $Cx=0$ except when $x=0$. The r independent columns in C combine to give all n columns in A.

R is the $r\times n$ matrix that tells us how to produce the columns of A from the columns of C. R contains the columns of the $r\times r$ identity matrix in the columns corresponding to the columns of A that are also in C.

C has the same column space as A. R has the same row space as A.

Column j of A is equal to C times column j of R. Row i of A is equal to row i of C times R.

If all columns of A are independent, then $C = A$ and $R = I$.

We find R through elimination. We perform Gaussian elimination on A, then remove any rows of all zeros to obtain R.

The r columns of C are a basis for the column space of A. The r rows of R are a basis for the row space of A.

We can also multiply matrices by taking $AB = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} \begin{bmatrix} - & b_1^* & - \\ & \vdots & \\ - & b_n^* & - \end{bmatrix} = a_1 b_1^* + a_2 b_2^* + \ldots + a_n b_n^*$, where the matrices $a_k b_k^*$ are outer products of rank 1.

R is the row reduced echelon form of A.