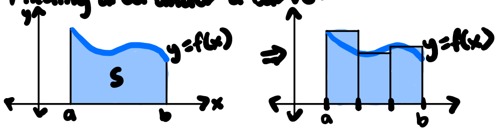
Chp 5: Integrals

5.1 Areas and Distances

Finding area under a curve:



The area A of the region S that lies under the graph of the continuous fn f is the limit of the sum of the areas of approximating rectangles: $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x]$.

We divide the interval $[a,b]$ into n subintervals, then choose a sample point (usually the left endpoint, right endpoint, or midpoint) within each subinterval. The y-value that corresponds to this sample point represents the height of the approximating rectangle.

Distance is the area under the velocity curve.

5.2 The Definite Integral

Constructing a Riemann sum: (finding approx area under f between x=a and x=b)

- We start with a function f defined on $[a,b]$.
- Divide $[a,b]$ into n smaller subintervals by choosing partition points $x_0, x_1, \ldots x_n$ s.t. $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$.
- The resulting collection of subintervals $[x_0,x_1], [x_1,x_2], [x_2,x_3], \ldots, [x_{n-1},x_n]$ is called a partition P of $[a,b]$.
- We use the notation Δx_i for the length of the i^{th} subinterval $[x_{i-1},x_i]$. Thus $\Delta x_i = x_i - x_{i-1}$.
- We choose sample points $x_1^*, x_2^*, \ldots, x_n^*$ with x_i^* in the i^{th} subinterval $[x_{i-1},x_i]$. These sample points could be left endpoints, right endpoints, midpoints, or any other numbers in the subintervals.
- The Riemann sum is $\sum_{i=1}^{n} f(x_i^*)\Delta x_i = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \ldots + f(x_n^*)\Delta x_n$.

If f is a fn defined on $[a,b]$, the definite integral of f from a to b is $\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$, provided that this limit exists. If it does exist, we say that f is integrable on $[a,b]$.

Thm: If f is cont on $[a,b]$, or if f has only a finite # of jump disconts, then f is integrable on $[a,b]$; that is, the definite integral $\int_a^b f(x)dx$ exists.

Thm: If f is integrable on $[a,b]$, then $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

Summation Formulas:

- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$
- $\sum_{i=1}^{n} c = nc$ for constant c
- $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
- $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
- $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

Midpoint Rule: $\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \Delta x[f(\bar{x}_1) + \ldots + f(\bar{x}_n)]$ where $\Delta x = \frac{b-a}{n}$ and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) =$ midpoint of $[x_{i-1}, x_i]$.

$\int_b^a f(x)dx = -\int_a^b f(x)dx$.

$\int_a^a f(x)dx = 0$.

Properties of the Integral:

1) $\int_a^b c\,dx = c(b-a)$ for constant c.
2) $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.
3) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ for constant c.
4) $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$.
5) $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$.
6) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$.
7) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
8) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

5.3 Evaluating Definite Integrals

Evaluation Thm: If f is cont on the interval $[a,b]$, then $\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f ($F' = f$).

$\int f(x)dx$ is an indefinite integral. $\int f(x)dx = F(x)$ means $F'(x) = f(x)$.

- Remember to add +C when integrating indefinite integrals!

Table of Indefinite Integrals:

$\int[f(x)+g(x)]dx=\int f(x)dx+\int g(x)dx$ | $\int cf(x)dx=c\int f(x)dx$ | $\int x^n dx=\frac{x^{n+1}}{n+1}+C\ (n\neq-1)$

$\int\frac{1}{x}dx=\ln(|x|)+C$ | $\int e^x dx=e^x+C$ | $\int a^x dx=\frac{a^x}{\ln a}+C$

$\int\sin x\,dx=-\cos x+C$ | $\int\cos x\,dx=\sin x+C$ | $\int\sec^2x\,dx=\tan x+C$

$\int\csc^2x\,dx=-\cot x+C$ | $\int\sec x\tan x\,dx=\sec x+C$ | $\int\csc x\cot x\,dx=-\csc x+C$

$\int\frac{1}{x^2+1}dx=\tan^{-1}x+C$ | $\int\frac{1}{\sqrt{1-x^2}}dx=\sin^{-1}x+C$ | $\int\sinh x\,dx=\cosh x+C$

$\int\cosh x\,dx=\sinh x+C$

Net Change Thm: The integral of a rate of change is the net change: $\int_a^b F'(x)dx=F(b)-F(a)$.

5.4 The Fundamental Theorem of Calculus

FTC1: If f is cont on $[a,b]$, then the fn g defined by $g(x)=\int_a^x f(t)dt$, $a\leq x\leq b$ is an antiderivative of f, that is, $g'(x)=f(x)$ for $a<x<b$.

FTC2: If f is cont on $[a,b]$, then $\int_a^b f(x)dx=F(b)-F(a)$, where F is any antiderivative of f, that is, $F'=f$.

The average value of f on the interval $[a,b]$ is $f_{avg}=\frac{1}{b-a}\int_a^b f(x)dx$.

The Mean Value Thm for Integrals: If f is cont on $[a,b]$, then $\exists c\in[a,b]$ s.t. $f(c)=f_{avg}=\frac{1}{b-a}\int_a^b f(x)dx$, that is, $\int_a^b f(x)dx=f(c)(b-a)$.