Chp 4: Applications of Differentiation

4.1 Maximum and Minimum Values

Let c be a number in the domain D of a function f. Then $f(c)$ is the absolute maximum value of f on D if $f(c) \geq f(x), \forall x \in D$, and $f(c)$ is the absolute minimum value of f on D if $f(c) \leq f(x), \forall x \in D$.

An absolute max/min is sometimes called a global max/min. The max & min values of f are called extreme values of f.

$f(c)$ is a local maximum of f if $f(c) \geq f(x)$ for x near c. $f(c)$ is a local minimum of f if $f(c) \leq f(x)$ for x near c.

Extreme Value Thm: If f is cont on a closed interval $[a,b]$, then f attains an absolute max value $f(c)$ and an absolute min value $f(d)$ at some $c, d \in [a,b]$.

Fermat's Thm: If f has a local max or min at c, and if $f'(c)$ exists, then $f'(c) = 0$.

A critical number of a fn f is a number c in the domain of f s.t. either $f'(c) = 0$ or $f'(c)$ DNE.

Thm: If f has a local max or min at c, then c is a critical number of f.

The Closed Interval Method: To find the absolute max & min values of a cont fn f on a closed interval $[a,b]$:

1) Find the values of f at the critical numbers of f in (a,b).
2) Find the values of f at the endpoints of the interval.
3) The largest of these values is the absolute max, and the smallest is the absolute min.

4.2 The Mean Value Theorem

Rolle's Thm: Let f be cont on $[a,b]$ and diff'ble on (a,b) s.t. $f(a) = f(b)$. Then $\exists c \in (a,b)$ s.t. $f'(c) = 0$.

Mean Value Thm: Let f be cont on $[a,b]$ and diff'ble on (a,b). Then $\exists c \in (a,b)$ s.t. $f'(c) = \frac{f(b) - f(a)}{b-a}$.

Thm: If $f'(x) = 0, \forall x \in (a,b)$, then f is constant on (a,b).

Cor: If $f'(x) = g'(x), \forall x \in (a,b)$, then $f - g$ is constant on (a,b); that is, $f(x) = g(x) + c$ for some constant c.

4.3 Derivatives and the Shapes of Graphs

Increasing/Decreasing Test:

a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.
b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

The First Derivative Test: Suppose that c is a critical number of a cont fn f.

a) If f' changes from positive to negative at c, then f has a local max at c.
b) If f' changes from negative to positive at c, then f has a local min at c.
c) If f' doesn't change sign at c, then f has no local max or min at c.

If the graph of f lies above all of its tangents on an interval I, then f is concave upward on I. If it lies below all of its tangents on I, then f is concave downward on I. (CC↑ and CC↓)

A point P on a curve $y = f(x)$ is called an inflection point of f if f is cont there and the curve changes from CC↑ to CC↓ or CC↓ to CC↑ at P.

Concavity Test:

a) If $f''(x) > 0$ for all $x \in I$, then f is CC↑ on I.
b) If $f''(x) < 0$ for all $x \in I$, then f is CC↓ on I.

The Second Derivative Test: Suppose f'' is cont near c.

a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min at c.
b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local max at c.

4.4 Curve Sketching

A) Domain: the set D of values of x for which $f(x)$ is defined.

B) Intercepts: The y-intercept is $f(0)$. To find the x-intercepts, set $y = 0$ and solve for x.

C) Symmetry:

i) If $f(-x) = f(x), \forall x \in D$, then f is even, so its graph is symmetric about the y-axis.
ii) If $f(-x) = -f(x), \forall x \in D$, then f is odd, so its graph is symmetric about the origin.
iii) If $f(x+p) = f(x), \forall x \in D$, where $p > 0$, then f is called a periodic function with period p.

D) Asymptotes:

i) The line $y = L$ is a horizontal asymptote if $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$.
ii) The line $x = a$ is a vertical asymptote if $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$.

E) Increasing/Decreasing: Use the Increasing/Decreasing Test.

F) Local Max/Min Values: Find the critical numbers of f, then use the First or Second Derivative Test.

G) Concavity & Inflection Points: Find the inflection points of f (where $f''(x) = 0$), then use the Concavity Test.

4.5 Optimization Problems

Steps:

1) Understand the problem.
2) Draw a diagram.
3) Introduce notation. Let Q be the quantity to be maximized/minimized, and assign symbols to other unknowns.
4) Express Q in terms of these unknowns.
5) Rewrite Q in terms of one variable.
6) Find the absolute max or min value of Q.

First Derivative Test for Absolute Extreme Values: Let c be a critical number of a cont fn f defined on an interval.

a) If $f'(x)>0$ for all $x<c$ and $f'(x)<0$ for all $x>c$, then $f(c)$ is the absolute max value of f.
b) If $f'(x)<0$ for all $x<c$ and $f'(x)>0$ for all $x>c$, then $f(c)$ is the absolute min value of f.

Revenue function for selling x units at price $p(x)$ is $R(x)=xp(x)$.
Profit function is $P(x)=R(x)-C(x)$.

4.7 Antiderivatives

A fn F is called an **antiderivative** of f on an interval I if $F'(x)=f(x)$, $\forall x\in I$.
Thm: If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x)+C$ for constant C.
Formulas:

Function	Antiderivative
$cf(x)$	$cF(x)$
$f(x)+g(x)$	$F(x)+G(x)$
x^n $(n\neq-1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x\tan x$	$\sec x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}x$
$\frac{1}{1+x^2}$	$\tan^{-1}x$
$\cosh x$	$\sinh x$
$\sinh x$	$\cosh x$