Chp 3: Inverse Functions

3.1 Exponential Functions

An exponential function is a fn of the form $f(x)=a^x$, where a is a positive constant.

Thm: If $a>0$ and $a\neq 1$, then $f(x)=a^x$ is a continuous fn w/ domain $\mathbb{R}$ and range $(0,\infty)$. In particular, $a^x>0$ for all x.

If $a,b>0$ and $x,y\in\mathbb{R}$, then:
1) $a^{x+y}=a^x a^y$
2) $a^{x-y}=\frac{a^x}{a^y}$
3) $(a^x)^y=a^{xy}$
4) $(ab)^x=a^x b^x$

If $a>1$, then $\lim_{x\to\infty}a^x=\infty$ and $\lim_{x\to-\infty}a^x=0$.

If $0<a<1$, then $\lim_{x\to\infty}a^x=0$ and $\lim_{x\to-\infty}a^x=\infty$.

$e=\lim_{x\to 0}(1+x)^{1/x}\approx 2.718$.

Natural exponential function is $y=e^x$.

$f(x)=e^x$ is a continuous fn w/ domain $\mathbb{R}$ & range $(0,\infty)$. Thus $e^x>0$ for all x. Also $\lim_{x\to-\infty}e^x=0$ and $\lim_{x\to\infty}e^x=\infty$, so the x-axis is a horizontal asymptote of $f(x)=e^x$.

3.2 Inverse Functions and Logarithms

A fn f is one-to-one (injective) if $x_1\neq x_2 \Rightarrow f(x_1)\neq f(x_2)$.

Horizontal Line Test: A fn is one-to-one IFF no horizontal line intersects its graph more than once.

Let $f:A\to B$ be a one-to-one fn. Then its inverse function $f^{-1}:B\to A$ is defined by $f^{-1}(y)=x \Leftrightarrow f(x)=y$.

Cancellation Equations: $\forall x\in A$, $f^{-1}(f(x))=x$, and $\forall x\in B$, $f(f^{-1}(x))=x$.

How to find inverse: Solve $y=f(x)$ for x, then interchange x & y.

Thm: If f is a one-to-one continuous fn, then its inverse f^{-1} is also cont.

Thm: If f is a one-to-one diff'ble fn w/ inverse f^{-1} and $f'(f^{-1}(a))\neq 0$, then f^{-1} is diff'ble at a and $(f^{-1})'(a)=\frac{1}{f'(f^{-1}(a))}$.

The logarithmic function w/ base a is $\log_a$, defined by $\log_a x=y \Leftrightarrow a^y=x$.

$\forall x\in\mathbb{R}$, $\log_a(a^x)=x$, and $\forall x>0$, $a^{\log_a x}=x$.

$\log_a$ has domain $(0,\infty)$ and range $\mathbb{R}$, and is cont.

Laws of Logarithms: If x & y are positive, then:
1) $\log_a(xy)=\log_a x+\log_a y$
2) $\log_a\left(\frac{x}{y}\right)=\log_a x-\log_a y$
3) $\forall r\in\mathbb{R}$, $\log_a(x^r)=r\log_a x$

If $a>1$, then $\lim_{x\to\infty}\log_a x=\infty$ and $\lim_{x\to 0^+}\log_a x=-\infty$.

The natural logarithm $\ln x=\log_e x$ is defined by $\ln x=y \Leftrightarrow e^y=x$.

$\forall x\in\mathbb{R}$, $\ln(e^x)=x$, and $\forall x>0$, $e^{\ln x}=x$. $\ln e=1$.

Change of Base Formula: $\forall a>0$, $a\neq 1$, we have $\log_a x=\frac{\ln x}{\ln a}$.

3.3 Derivatives of Log & Exp Functions

Thm: The fn $f(x)=\log_a x$ is diff'ble and $f'(x)=\frac{1}{x}\log_a e=\frac{1}{x\ln a}$.

$\frac{d}{dx}(\ln x)=\frac{1}{x}=\frac{d}{dx}(\ln|x|)$.

Steps in Log Differentiation:
1) Take natural logs of both sides of $y=f(x)$ and use the laws of logs to simplify.
2) Differentiate implicitly w.r.t x.

3) Solve for y'.

Thm: $f(x)=a^x$, $a>0$ is diff'ble, and $\frac{d}{dx}(a^x)=a^x \ln a$.

$\frac{d}{dx}(e^x)=e^x$.

3.4 Exponential Growth and Decay

Law of Natural Growth/Decay: If $y(t)$ is the value of y at time t and $\frac{dy}{dt}$ is proportional to $y(t)$, then $\frac{dy}{dt}=ky$.

Thm: The only solutions of the diff eq $\frac{dy}{dt}=ky$ are the exponential fns $y(t)=y(0)e^{kt}$.

Relative growth rate of a pop is $\frac{1}{P}\cdot\frac{dP}{dt}$.

Newton's Law of Cooling: $\frac{dT}{dt}=k(T-T_s)$ where $T(t)$ is temp of object at time t and T_s is temp of surroundings.

3.5 Inverse Trig Functions

$\arcsin x=\sin^{-1}x=y \Leftrightarrow \sin y=x$ and $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$.

$\sin^{-1}(\sin x)=x$ for $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$ and $\sin(\sin^{-1}x)=x$ for $-1\le x\le 1$.

$\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$, $-1<x<1$.

$\arccos x=\cos^{-1}x=y \Leftrightarrow \cos y=x$ and $0\le y\le\pi$.

$\cos^{-1}(\cos x)=x$ for $0\le x\le\pi$ and $\cos(\cos^{-1}x)=x$ for $-1\le x\le 1$.

$\frac{d}{dx}(\cos^{-1}x)=-\frac{1}{\sqrt{1-x^2}}$, $-1<x<1$.

$\arctan x=\tan^{-1}x=y \Leftrightarrow \tan y=x$ and $-\frac{\pi}{2}<y<\frac{\pi}{2}$.

$\lim_{x\to\infty}\tan^{-1}x=\frac{\pi}{2}$ and $\lim_{x\to-\infty}\tan^{-1}x=-\frac{\pi}{2}$.

$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$.

$\csc^{-1}x=y$ $(|x|\ge 1) \Leftrightarrow \csc y=x$ and $y\in(0,\frac{\pi}{2}]\cup(\pi,\frac{3\pi}{2}]$.

$\sec^{-1}x=y$ $(|x|\ge 1) \Leftrightarrow \sec y=x$ and $y\in[0,\frac{\pi}{2})\cup[\pi,\frac{3\pi}{2})$.

$\cot^{-1}x=y$ $(x\in\mathbb{R}) \Leftrightarrow \cot y=x$ and $y\in(0,\pi)$.

Function	Derivative
$\sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1}x$	$\frac{1}{1+x^2}$
$\csc^{-1}x$	$-\frac{1}{x\sqrt{x^2-1}}$
$\sec^{-1}x$	$\frac{1}{x\sqrt{x^2-1}}$
$\cot^{-1}x$	$-\frac{1}{1+x^2}$

3.7 Indeterminate Forms & L'Hospital's Rule

L'Hospital's Rule: If $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$ or $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$, then if f & g are diff'ble and $g'(x)\neq 0$, then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$.