Chp 2: Derivatives

2.1 Derivatives and Rate of Change

The tangent line to $y=f(x)$ at the point $P(a,f(a))$ is the line through P w/ slope $m=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.

Ex: Find an equation of the tangent line to $y=\frac{3}{x}$ at $(3,1)$.

$m=\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0}\frac{\frac{3}{3+h}-1}{h}=\lim_{h\to 0}\frac{3-(3+h)}{h(3+h)}=\lim_{h\to 0}\frac{-h}{h(3+h)}=\lim_{h\to 0}\frac{-1}{3+h}=-\frac{1}{3}$.

$y-1=-\frac{1}{3}(x-3)\Rightarrow y-1=-\frac{1}{3}x+1\Rightarrow y=-\frac{1}{3}x+2$

The derivative of a function f at a is $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.

2.2 The Derivative as a Function

The derivative of f is the function $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

A function f is differentiable at a if $f'(a)$ exists. It's diff'ble on (a,b) if it's diff'ble for every $c\in(a,b)$.

Thm: If f is diff'ble at a, then f is cont at a.

2.3 Basic Differentiation Formulas

$\frac{d}{dx}(c)=0$ for a constant c.

$\frac{d}{dx}(x)=1$.

Power Rule: if $n\in\mathbb{R}$, then $\frac{d}{dx}(x^n)=nx^{n-1}$.

Constant Multiple Rule: $\frac{d}{dx}[cf(x)]=c\frac{d}{dx}f(x)$.

Sum Rule: $\frac{d}{dx}[f(x)+g(x)]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)$.

Difference Rule: $\frac{d}{dx}[f(x)-g(x)]=\frac{d}{dx}f(x)-\frac{d}{dx}g(x)$.

$\frac{d}{dx}(\sin x)=\cos x$.

$\frac{d}{dx}(\cos x)=-\sin x$.

2.4 The Product and Quotient Rules

Product Rule: $\frac{d}{dx}[f(x)g(x)]=f(x)g'(x)+f'(x)g(x)$.

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$.

$\frac{d}{dx}(\tan x)=\sec^2 x$.

$\frac{d}{dx}(\csc x)=-\csc x\cot x$.

$\frac{d}{dx}(\sec x)=\sec x\tan x$.

$\frac{d}{dx}(\cot x)=-\csc^2 x$.

2.5 The Chain Rule

Chain Rule: If $F(x)=f(g(x))$, then $F'(x)=f'(g(x))\cdot g'(x)$.

2.6 Implicit Differentiation

Ex: If $x^2+y^2=25$, find $\frac{dy}{dx}$.

$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(25)$.

$2x+2y\frac{dy}{dx}=0$.

$2y\frac{dy}{dx}=-2x$.

$\frac{dy}{dx}=-\frac{2x}{2y}=-\frac{x}{y}$.

2.7 Related Rates

Ex: Air is pumped into a spherical balloon s.t. its volume increases by 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

$\frac{dV}{dt} = 100\ cm^3/s$, $V = \frac{4}{3}\pi r^3$, want to find $\frac{dr}{dt}$ when $2r = 50 \Rightarrow r = 25$.

$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$.

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$.

$\frac{dr}{dt} = \frac{1}{4\pi r^2} \cdot \frac{dV}{dt} = \frac{1}{4\pi r^2} \cdot 100 = \frac{25}{\pi r^2}$.

At $r = 25$, $\frac{dr}{dt} = \frac{25}{\pi (25)^2} = \frac{1}{25\pi}$.

2.8 Linear Approximation and Differentials

The linear approximation of f at a is $f(x) \approx f(a) + f'(a)(x-a)$.