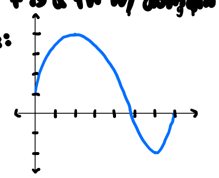
Chp 1: Functions & Limits

1.1 Functions & Their Representations

A **function** f is a rule that assigns to each element $x \in D$ exactly one element $f(x) \in E$. The set D is the **domain** of the function.
The **range** of f is the set of all possible values of $f(x)$.

An **independent variable** is a symbol that represents an arbitrary number in the domain of a fn f. A **dependent var** is a symbol that represents a # in the range of f.

If f is a fn w/ domain D then its **graph** is the set of ordered pairs $\{(x, f(x)) \mid x \in D\}$.

Ex:


a) Find the values of $f(1)$ and $f(5)$.
$f(1) = 3$, $f(5) = -1$

b) What are the domain & range of f?
Domain: $x \in [0, 7]$, range: $f(x) \in [-2, 4]$

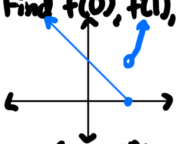
Four ways to represent a fn: verbally, numerically (table), visually (graph), algebraically (formula)

Vertical line test: a curve in the xy-plane is the graph of a fn of x IFF no vertical line intersects the curve more than once.

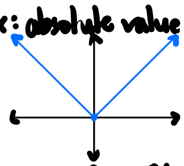
Piecewise functions are defined by diff formulas in diff parts of their domains.

Ex: $f(x) = \begin{cases} 1-x, & x \leq 1; \\ x^2, & x > 1. \end{cases}$ Find $f(0)$, $f(1)$, & $f(2)$ and sketch the graph.

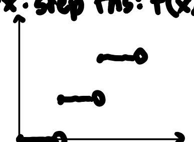
$f(0) = 1 - 0 = 1.$
$f(1) = 1 - 1 = 0.$
$f(2) = 2^2 = 4.$



Ex: absolute value: $|a| = \begin{cases} a, & a \geq 0; \\ -a, & a < 0. \end{cases}$ Sketch the graph of $f(x) = |x|$.



Ex: step fns: $f(x) = \lfloor x \rfloor$



A function f is an **even function** if $f(-x) = f(x)$ for every number x in its domain. Graph symmetric w.r.t. y-axis.

Ex: $f(x) = x^2$ is even b/c $f(-x) = (-x)^2 = x^2 = f(x)$.

A function f is an **odd function** if $f(-x) = -f(x)$ for every x in its domain. Graph symmetric about the origin.

Ex: $f(x) = x^3$ is odd b/c $f(-x) = (-x)^3 = -x^3 = -f(x)$.

Ex: Determine whether each is even, odd, or neither.

a) $f(x) = x^5 + x$.
$f(-x) = (-x)^5 + (-x) = -x^5 - x = -f(x)$. Odd.

b) $g(x) = 1 - x^4$.
$g(-x) = 1 - (-x^4) = 1 - x^4 = g(x)$. Even.

c) $h(x) = 2x - x^2$.
$h(-x) = 2(-x) - (-x)^2 = -2x - x^2$. Neither.

A function f is **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

A function f is **decreasing** on an interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.

1.2 A Catalog of Essential Functions

Mathematical model - a mathematical description of a real-world phenomenon. Idealization. Steps:

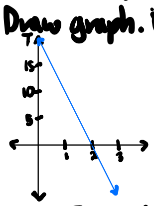
1) Identify & name indep & dep vars, make assumptions to simplify phenomenon. Obtain eqs to relate vars (Formulate)
2) Solve: apply math to the model to derive mathematical conclusions.
3) Interpret: interpret conclusions as info about the real-world phenomenon.
4) Test: check our predictions against new real data. If they don't compare well, we need to refine or reformulate our model.

If y is a **linear function** of x, then the graph is a line, so we can use slope-intercept form to write it as $y=f(x)=mx+b$.

Ex: a) As dry air moves upward, it expands & cools. If ground temp is $20°C$ & temp at height of 1 km is $10°C$, express the temp T (°C) as a fn of the height h (km), assuming that a linear model is appropriate.

We have 2 points: $(0,20)$ and $(1,10)$. Slope: $m=\frac{20-10}{0-1}=\frac{10}{-1}=-10$. So $T-20=-10(h-0) \Rightarrow T=-10h+20$.

b) Draw graph. What does the slope represent?



Slope of $m=-10°C/km$ represents the rate of change of temp w.r.t. time.

c) What is the temp at a height of 2.5 km?

$T(2.5)=-10(2.5)+20=-25+20=-5\ °C$.

A function P is called a **polynomial** if $P(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0$, where $n\in\mathbb{N}$ and $a_0,\ldots,a_n$ are constant **coefficients**.

- The domain of any polynomial is $\mathbb{R}$.
- If the leading coeff $a_n\neq 0$, then the **degree** of the polynomial is n.

Ex: $P(x)=2x^6-x^4+\frac{2}{5}x^3+\sqrt{2}$ is a polynomial of degree 6.

A linear function is a polynomial of degree 1.

A **quadratic function** is a polynomial of degree 2, w/ form $P(x)=ax^2+bx+c$. Its graph is a parabola obtained by shifting the parabola $y=ax^2$. The parabola opens upward if $a>0$ and downward if $a<0$.

A **cubic function** is a polynomial of degree 3, w/ form $P(x)=ax^3+bx^2+cx+d$.

A **power function** is a fn of the form $f(x)=x^a$, where a is a constant. Cases:

i) $a=n, n\in\mathbb{Z}^+$.
ii) $a=1/n, n\in\mathbb{Z}^+$.
iii) $a=-1$ (reciprocal fn)

A **rational function** f is a ratio of two polynomials: $f(x)=\frac{P(x)}{Q(x)}$.

- Domain consists of all values of x s.t. $Q(x)\neq 0$.

Trigonometric functions involve sin, cos, tan, csc, sec, cot, or their inverses.

Both sin & cos have domain $\mathbb{R}$ and range $[-1,1]$. So $\forall x$, $-1\le\sin(x)\le 1$ and $-1\le\cos(x)\le 1$, or $|\sin x|\le 1$ and $|\cos x|\le 1$.

The zeroes of sine occur at the integer multiples of π, so $\sin x=0$ when $x=n\pi$, $n\in\mathbb{Z}$.

Sine and cosine are **periodic** fns w/ period 2π, so $\forall x$, $\sin(x+2\pi)=\sin(x)$ and $\cos(x+2\pi)=\cos(x)$.

Tangent fn: $\tan x=\frac{\sin x}{\cos x}$, undefined when $\cos x=0$ ($x=\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\ldots$). Range is $\mathbb{R}$. Period is π: $\tan(x+\pi)=\tan(x)$, $\forall x$.

Cosecant, secant, and cotangent are reciprocals.

Exponential functions have form $f(x)=a^x$, where a is a positive constant.

Inverse of exponential fns are **logarithmic functions** $f(x)=\log_a(x)$, where the base a is a positive constant.

Translations: If c is positive, then $y=f(x)+c$ is just $y=f(x)$ shifted up by c units, and $y=f(x-c)$ is $y=f(x)$ shifted right by c.

Stretching: If $c>1$, then $y=cf(x)$ is $y=f(x)$ stretched vertically by a factor of c.

Reflecting: $y=-f(x)$ is $y=f(x)$ reflected about the x-axis.

Sum/diff of fns: $(f+g)(x)=f(x)+g(x)$ and $(f-g)(x)=f(x)-g(x)$.

- If f has domain A & g has domain B, then $f+g$ has domain $A\cap B$.

Product/quotient of fns: $(fg)(x)=f(x)g(x)$ and $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$.
- Domain of fg is $A\cap B$
- Domain of $\frac{f}{g}$ is $\{x\in A\cap B \mid g(x)\neq 0\}$.

Given two functions f and g, their composition is defined by $(f\circ g)(x)=f(g(x))$.
- Domain of $f\circ g$ is the set of all x in the domain of g s.t. $g(x)$ is in the domain of f.

1.3 The Limit of a Function

Def: Suppose $f(x)$ is defined when x is near a. Then the limit $\lim_{x\to a} f(x)=L$ if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to, but not equal to, a.

We write $\lim_{x\to a^-} f(x)=L$ & say the left-hand limit equals L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and $x<a$.

Def: Let f be a fn defined on some open interval that contains the number a, except possibly a itself. Then the limit $\lim_{x\to a} f(x)=L$ if $\forall \varepsilon>0, \exists \delta>0$ s.t. $0<|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon$.

1.4 Calculating Limits

Limit laws: Suppose c is a constant & $\lim_{x\to a} f(x)$ & $\lim_{x\to a} g(x)$ exist. Then:

1) $\lim_{x\to a}[f(x)+g(x)]=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)$.
2) $\lim_{x\to a}[f(x)-g(x)]=\lim_{x\to a} f(x)-\lim_{x\to a} g(x)$.
3) $\lim_{x\to a}[cf(x)]=c\lim_{x\to a} f(x)$.
4) $\lim_{x\to a}[f(x)g(x)]=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$.
5) $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x)\neq 0$.
6) $\lim_{x\to a}[f(x)]^n=[\lim_{x\to a} f(x)]^n$, where $n\in\mathbb{Z}^+$.
7) $\lim_{x\to a} c=c$.
8) $\lim_{x\to a} x=a$.
9) $\lim_{x\to a} x^n=a^n$, where $n\in\mathbb{Z}^+$.
10) $\lim_{x\to a}\sqrt[n]{x}=\sqrt[n]{a}$, where $n\in\mathbb{Z}^+$.
11) $\lim_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to a} f(x)}$, where $n\in\mathbb{Z}^+$.

Direct Substitution Property: if f is a polynomial or a rational fn and a is in the domain of f, then $\lim_{x\to a} f(x)=f(a)$.

Thm: $\lim_{x\to a} f(x)=L$ IFF $\lim_{x\to a^-} f(x)=L=\lim_{x\to a^+} f(x)$.

Thm: If $f(x)\leq g(x)$ when x is near a (except possibly at a) and the limits of f & g both exist as $x\to a$, then $\lim_{x\to a} f(x)\leq\lim_{x\to a} g(x)$.

Squeeze Thm: if $f(x)\leq g(x)\leq h(x)$ when x is near a (except possibly at a) and $\lim_{x\to a} f(x)=\lim_{x\to a} h(x)=L$, then $\lim_{x\to a} g(x)=L$.

1.5 Continuity

A function f is continuous at a if $\lim_{x\to a} f(x)=f(a)$.

3 implicit requirements:
1) $f(a)$ is defined (so a is in the domain of f).
2) $\lim_{x\to a} f(x)$ exists.
3) $\lim_{x\to a} f(x)=f(a)$.

A fn f is cont from the right at a if $\lim_{x\to a^+} f(x)=f(a)$ and from the left if $\lim_{x\to a^-} f(x)=f(a)$.

A fn f is cont on an interval if it's cont at every number in the interval.

Thm: If f & g are cont at a and c is a constant, then the following fns are also cont at a:
1) $f+g$
2) $f-g$

3) cf
4) fg
5) $\frac{f}{g}$ if $g(a) \neq 0$.

Thm: a) Any polynomial is cont everywhere (on $\mathbb{R}$).
b) Any rational fn is cont wherever it's defined (on its domain).

Thm: Polynomials, rational fns, root fns, and trig fns are cont at every # in their domain.

Thm: If f is cont at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$. In other words, $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$.

Thm: If g is cont at a and f is cont at $g(a)$, then $f\circ g$ is cont at a.

Intermediate Value Thm: Suppose that f is cont on $[a,b]$ and let N be any # b/w $f(a)$ & $f(b)$, where $f(a) \neq f(b)$. Then $\exists c \in (a,b)$ s.t. $f(c) = N$.

1.6 Limits Involving Infinity

$\lim_{x\to a} f(x) = \infty$ means that $f(x)$ approaches ∞ as $x \to a$.

The line $x=a$ is a vertical asymptote of $y=f(x)$ if at least one of the following is true: $\lim_{x\to a} f(x) = \infty$, $\lim_{x\to a^-} f(x) = \infty$, $\lim_{x\to a^+} f(x) = \infty$, $\lim_{x\to a} f(x) = -\infty$, $\lim_{x\to a^-} f(x) = -\infty$, $\lim_{x\to a^+} f(x) = -\infty$.

The line $y=L$ is a horizontal asymptote if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to -\infty} f(x) = L$.

If n is a positive integer, then $\lim_{x\to\infty} \frac{1}{x^n} = 0$ and $\lim_{x\to -\infty} \frac{1}{x^n} = 0$.

Def: let f be a fn defined on some open interval containing a, except possibly a itself. Then $\lim_{x\to a} f(x) = \infty$ means that for every positive # M there's a positive # δ s.t. $0 < |x-a| < \delta \Rightarrow f(x) > M$.

Def: let f be defined on some interval (a, ∞). Then $\lim_{x\to\infty} f(x) = L$ means that $\forall \varepsilon > 0$, $\exists N$ s.t. $x > N \Rightarrow |f(x) - L| < \varepsilon$.

Def: Let f be defined on some interval (a, ∞). Then $\lim_{x\to\infty} f(x) = \infty$ means that $\forall M > 0$, $\exists N$ s.t. $x > N \Rightarrow f(x) > M$.